ANALYTICAL MECHANICS

FOR STUDENTS OF PHYSICS AND ENGINEERING

BY

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PREFACE.

THE following work is based upon a course of lectures and recitations which the author has given, during the last few years, to the Junior class of the Electrical Engineering Department of the Sheffield Scientific School.

It has been the author's aim to present the subject in such a manner as to enable the student to acquire a firm grasp of the fundamental principles of Mechanics and to apply them to problems with the minimum amount of mental effort. In other words *economy of thought* is the goal at which the author has aimed. It should not be understood, however, that the author has been led by the tendency toward reducing text-books to collections of rules, mnemonic forms, and formulæ. Rules and drill methods tend toward the exclusion of reasoning rather than toward efficiency in thinking. The following features of the treatment of the subject may be noted:

In order to make the book suitable for the purposes of more than one class of students more special topics are discussed than any one class will probably take up. But these are so arranged as to permit the omission of one or more without breaking the logical continuity of the subject.

In deciding on the order of the topics discussed two more or less conflicting factors have been kept before the eye, i.e., to make the treatment logical, yet to introduce as few new concepts at a time as possible. It is to secure the second of these ends, for instance, that the historical order of the development of mechanics is followed by discussing equilibrium before motion. This arrangement not only

grades the path of the student by leading him from the easier to the more difficult dynamical ideas, but it also gives him time to acquire proficiency in the use of his mathematical tools.

As a result of the severe criticisms of Newton's laws of motion by such men as Heinrich Hertz, Ernst Mach, and Karl Pearson, authors of recent text-books on Mechanics have shown a tendency to give either a new set of laws or none at all. There is no doubt that a subject like Mechanics should start, as in the case of Thermodynamics, with a few simple laws and the entire structure of the science should be based upon them. In the present work the following law is made the basis of the entire subject:

To every action there is an equal and opposite reaction, or, the sum of all the actions to which a body or a part of a body is subject at any instant vanishes.

Four concepts are associated with the term *action*, namely, the concepts of *force, torque, linear kinetic reaction,* and *angular kinetic reaction.* These are introduced one at a time and in connection with the application of the law to a certain class of problems. Force is introduced with the equilibrium of a particle (pp. 15, 16), torque with the equilibrium of a rigid body (pp. 35, 39, 40), linear kinetic reaction with the motion of a particle (pp. 100–106), angular kinetic reaction with the motion of a rigid body (pp. 218–221). Thus by introducing the concepts of *linear* and *angular reactions* and by extending the meaning of the term *action* to include these reactions, the fundamental principle of Mechanics is put in the form of a single law, which is equivalent to Newton's laws of motion and which has the advantages of the point of view involved in D'Alembert's principle. This law has the directness and simplicity of Newton's third law, so that the beginner can easily understand it and apply it to simple problems of equilibrium, and yet it admits of wider interpretation and application with the growth of

the student's knowledge. By making this law the central idea of the entire subject and by gradually extending its interpretation the treatment is made uniform, coherent, and progressive.

While appeal is made to the student's experience in introducing the principles of the conservation of dynamical energy and of the conservation of momentum they are shown to be direct consequences of the law of action and reaction. The equivalence and the alternative character of the conservation principles and of the law of action and reaction are emphasized by working out a number of problems by the application of both the law and the principles.

The two types of motion, i.e., motion of translation and motion of rotation, are treated not only in the same general manner, but are developed along almost parallel lines.

The simpler types of motion which are generally treated under Kinematics are given in the present work as problems in Dynamics. The author believes that the practice of divesting the physical character of the motion from the simpler types and reducing them to problems in integration is unfortunate. On account of their freedom from mathematical difficulties the simpler types of motion are particularly well adapted to illustrate the principles of dynamics.

In order to differentiate between vectors and their magnitudes the former are printed in the Gothic type.

In conclusion the author wishes to express his obligations to Mr. Leigh Page for reading the manuscript and to Dr. David D. Leib for reading the proofs and to both for many valuable suggestions.

H. M. Dadourian.

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TABLE OF NOTATIONS.

a = radius, length, constant.
b = radius, length, constant.
c = const., center of mass.
d = distance.
e = nap. base, coef. of restitution.
f = acceleration.
g = grav. acceleration.
h = vertical height, constant.
i = $\sqrt{-1}$.
k = constant.
l = length, direction cosine.
m = mass, direction cosine.
n = number, direction cosine.
o = origin, center.
p = pressure, page.
q = variable magnitude.
r = radius, radius vector.
s = strain, length of curve.
t = time.
u = velocity.
v = velocity, volume.
w = weight.
x = variable magnitude.
y = variable magnitude.
z = variable magnitude.

A = area, point, constant.
B = point, constant.
C = point, constant.
D = distance.
E = total energy.
F = force, frictional force.
G = moment of force or torque.
H = height, force derived from potential, angular impulse.
I = moment of inertia.
K = radius of gyration, constant.
L = length, linear impulse.
M = mass.
N = normal component of force.
O = origin, point.
P = period, point, power.
Q = point.
R = total reaction, resultant force.
S = stress.
T = tensile force, kinetic energy.
U = potential energy.
V = velocity, potential.
W = weight, work.
X = x-component of force.
Y = y-component of force.
Z = z-component of force.

$\alpha, \beta, \gamma, \delta$ = constant angles.
θ, ϕ, ψ = variable angles.
ϕ = angle of friction.
μ = coef. of friction.
λ = modulus of elasticity.
γ = angular acceleration.
ω = angular velocity.
ϵ = a small quantity, or angle.
σ = surface density.
τ = volume density.
ρ = linear density, radius of curvature.

"Σ" = "sum of all the . . . s."
"$\equiv$" = "is identical with."
"$\doteq$" = "approaches."
"$\doteqdot$" = "equals approximately."
"$<$" = "is smaller than."
"$\ll$" = "is very small compared with."
"$>$" = "is greater than."
"$\gg$" = "is very large compared with."
$\underline{n|}$ = n! = 1·2·3. . . . n.

lb. = pound, the unit of weight.	gm. = gram.
pd. = mass of a body which weighs one pound.	kg. = kilogram.
in. = inch.	mm. = millimeter.
ft. = foot.	cm. = centimeter.
H.P. = horse power.	m. = meter.
S.H.M. = simple harmonic motion.	km. = kilometer.

Letters in gothic type denote vector magnitudes.

A dot on a letter indicates that the latter is differentiated with respect to time.

A letter with a bar above it denotes an average magnitude.

The letter "i" when used as a subscript denotes "any one of . . . ," thus "F_i" stands for "any one of F_1, F_2, F_3, etc."

ANALYTICAL MECHANICS

INTRODUCTION.

1. Scope and Aim of Mechanics. — Mechanics is the science of motion. It has a twofold object:

First, to describe the motions of bodies and to interpret them by means of a few laws and principles, which are generalizations derived from observation and experience.

Second, to predict the motion of bodies for all times when the circumstances of the motion for any one instant are given, in addition to the special laws which govern the motion.

The present tendency in science is toward regarding all physical phenomena as manifestations of motion. Complicated and apparently dissimilar phenomena are being explained by the interactions and motions of electrons, atoms, molecules, cells, and other particles. The kinetic theory of heat, the wave theories of sound and light, and the electron theory of electricity are examples which illustrate the tendency toward a mechanical interpretation of the physical universe.

This tendency not only emphasizes the fundamental importance of the science of mechanics to other physical sciences and engineering but it also broadens the aim of the science and makes the dynamical interpretation of all physical phenomena its ultimate object. The aim of elementary mechanics is, however, very modest and its scope is limited to the discussion of the simplest cases of motion and equilibrium which occur in nature.

2. Divisions of Mechanics. — It is customary to divide Mechanics into *Kinematics* and *Dynamics*. The former treats of the time and space relations of the motions of bodies without regard to the interactions which cause them. In other words, Kinematics is the geometry of motion. In Dynamics, on the other hand, motion and equilibrium are treated as the results of interactions between bodies; consequently not only *time* and *space* enter into dynamical discussions, but also *mass*, the third element of motion.

Dynamics in its turn is divided into *Statics* and *Kinetics*. Statics is the mechanics of bodies in equilibrium, while Kinetics is the mechanics of bodies in motion.

Chapters II, III, and IV of the present work are devoted to problems in statics, while the rest of the book, with the exception of Chapters I, V, and VII, is given to discussions of problems in kinetics. The subject matter of Chapters I and VII is essentially of a mathematical nature. In the former the addition and resolution of vectors are discussed, while in the latter the Calculus is applied to finding centers of mass and moments of inertia. Chapter V is devoted mainly to kinematical problems.

CHAPTER I.

ADDITION AND RESOLUTION OF VECTORS.

3. Scalar and Vector Magnitudes.— Physical magnitudes may be divided into two classes according to whether they have the property of orientation or not. Magnitudes which have direction are called *vectors*, while those which do not have this property are called *scalars*. Displacement, velocity, acceleration, force, torque, and momentum are vector magnitudes. Mass, density, work, energy, and time are scalars.

4. Graphical Representation of Vectors. — Vectors are represented by directed lines or arrows. The length of the directed line represents the magnitude of the vector, while its direction coincides with that of the vector. For brevity the directed lines as well as the physical quantities which they represent are called vectors. The head and the tail of the directed line are called, respectively, the *terminus* and the *origin* of the vector. In Fig. 1, for instance, P is the origin and Q the terminus of the vector $\mathbf{a}$.

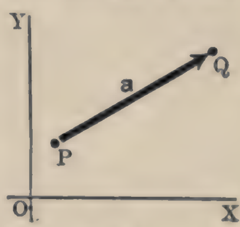


FIG. 1.

5. Notation. — Vectors will be denoted by letters printed in Gothic type, while their magnitudes will be represented by the same letters printed in italic type. Thus in Fig. 1 the vector PQ is denoted by $\mathbf{a}$, but if it is desired to represent the length PQ without regard to its orientation a is used.

6. Equal Vectors. — Two vectors are said to be equal if they have the same length and the same direction. It follows, therefore, that the value of a vector is not changed when it

is moved about without changing its direction and magnitude.

7. Addition of Two Vectors.— Let the vectors **a** and **b**, Fig. 2, represent two displacements, then their sum is another vector, **c**, which is equivalent to the given vectors. In order to find **c** let us apply to a particle the operations indicated by **a** and **b**. Each vector displaces the particle along its direction through a distance equal to its length. There-

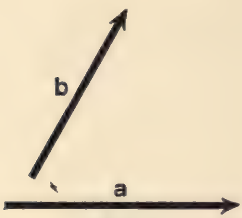


Fig. 2.

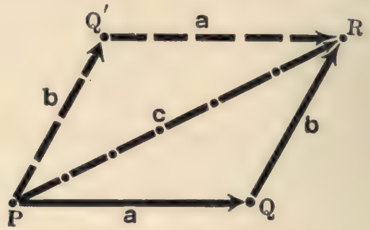


Fig. 3.

fore applying **a** to the particle at P, Fig. 3, the particle is brought to the point Q. Then applying the operation indicated by **b** the particle is brought to the point R. Therefore the result of the two operations is a displacement from P to R. But this is equivalent to a single operation represented by the vector **c**, which has P for its origin and R for its terminus. Therefore **c** is called the sum, or the resultant, of **a** and **b**. This fact is denoted by the following vector equation,

$$\mathbf{a} + \mathbf{b} = \mathbf{c}. \tag{I}$$

8. Order of Addition. — The order of addition does not affect the result. If in Fig. 3 the order of the operations indicated by **a** and **b** is reversed the particle moves from P to Q' and then to R. Thus the path of the particle is changed but not the resultant displacement.

9. Simultaneous Operation of Two Vectors.—The operations indicated by **a** and **b** may be performed simultaneously without affecting the final result. In order to illustrate

the simultaneous operation of two vectors suppose the particle to be a bead on the wire AB, Fig. 4. Move the wire, keeping it parallel to itself, until each of its particles is given a displacement represented by **b**. Simultaneously with the motion of the wire move the bead along the wire giving it a displacement equal to **a**. At the end of

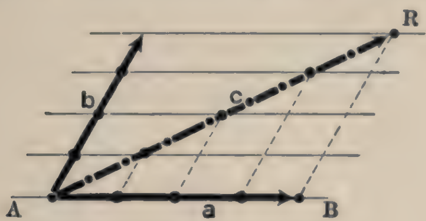


FIG. 4.

these operations the bead arrives at the point R. If both the wire and the bead are moved at constant rates the resultant vector **c** represents not only the resulting displacement but also the path of the particle.

10. Rules for Adding Two Vectors.—The results of the last three paragraphs furnish us with the following methods for adding two vectors graphically.

Triangle Method.—*Move one of the vectors, without changing its direction, until its origin falls upon the terminus of the other vector, then complete the triangle by drawing a vector the origin of which coincides with that of the first vector. The new vector is the resultant of the given vectors.*

Parallelogram Method.—*Move one of the vectors until its origin falls on that of the other vector, complete the parallelogram, and then draw a vector which has the common origin of the given vectors for its origin and which forms a diagonal of the parallelogram. The new vector is the resultant of the given vectors.*

11. Analytical Expression for the Resultant of Two Vectors.—Let **a** and **b**, Fig. 5, be two vectors and **c** their result-

ant. Then, solving the triangle formed by these vectors, we obtain

$$c^2 = a^2 + b^2 + 2\,ab \cos \phi \tag{II}$$

and

$$\tan \theta = \frac{b \sin \phi}{a + b \cos \phi}, \tag{III}$$

where a, b, and c are the magnitudes of **a**, **b**, and **c**, respectively, while ϕ and θ are the angles **b** and **c** make with **a**. Equation (II) determines the magnitude and equation (III) the direction of **c**.

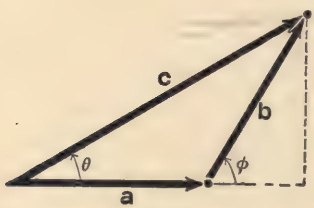


Fig. 5.

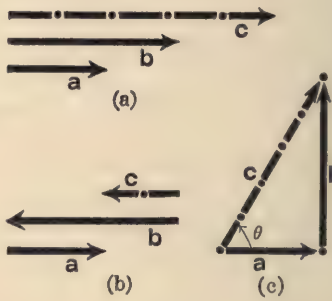


Fig. 6.

Special Cases. (a) If **a** and **b** have the same direction, as in Fig. 6a, then $\phi = 0$. Therefore

$$c^2 = a^2 + b^2 + 2\,ab, \qquad \therefore \quad c = a + b,$$

and

$$\tan \theta = 0, \qquad \therefore \quad \theta = 0.$$

Thus **c** has the same direction as **a** and **b**, while its magnitude equals the arithmetical sum of their magnitudes.

(b) When **a** and **b** are oppositely directed, as in Fig. 6b, $\phi = \pi$. Therefore

$$c^2 = a^2 + b^2 - 2\,ab, \qquad \therefore \quad c = a - b,$$

and

$$\tan \theta = 0, \qquad \therefore \quad \theta = 0.$$

Thus the magnitude of **c** equals the algebraic sum of the magnitudes of **a** and **b**, while its direction is the same as that of the larger of the two. It is evident that if the magnitudes of **a** and **b** are equal **c** vanishes. Therefore two vectors of equal magnitude and opposite directions are the negatives of each other. In other words, *when the direction of a vector is reversed its sign is changed.*

(c) When **a** and **b** are at right angles to each other, as in Fig. 6c, $\phi = \frac{\pi}{2}\cdot$ Therefore

$$c^2 = a^2 + b^2$$

and

$$\tan\theta = \frac{b}{a}\cdot$$

12. Difference of Two Vectors. — Subtraction is equivalent to the addition of a negative quantity. Therefore, to subtract **b** from **a** we add $-$**b** to **a**. Thus we have the following rule for subtracting one vector from another.

In order to subtract one vector from another reverse the one to be subtracted and add it to the other vector.

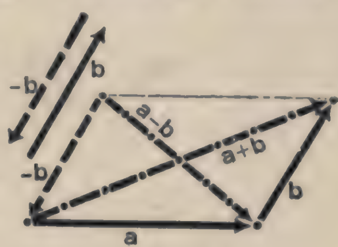


Fig. 7.

It is evident from Fig. 7 that the sum and the difference of two vectors form the diagonals of the parallelogram determined by them.

ILLUSTRATIVE EXAMPLES.

A particle is displaced 10 cm. N. 30° E., then 10 cm. E. Find the resulting displacement.

Representing the displacements and their resultant by the vectors **a**, **b**, and **c**, Fig. 8, we obtain

$$
\begin{aligned}
c^2 &= a^2 + b^2 + 2\,ab\cos\phi \\
&= (10\text{ cm.})^2 + (10\text{ cm.})^2 + 2 \times 10\text{ cm.} \times 10\text{ cm.}\cos(60^\circ) \\
&= 300\text{ cm.}^2 \\
\therefore \quad c &= 10\sqrt{3}\text{ cm.} \\
&\doteqdot 17.3\text{ cm.}^* \\
\tan\theta &= \frac{b\sin\phi}{a + b\cos\phi} \\
&= \frac{10\text{ cm.}\sin(60^\circ)}{10\text{ cm.} + 10\text{ cm.}\cos(60^\circ)} \\
&= \tfrac{1}{3}\sqrt{3}. \\
\therefore \quad \theta &= 30^\circ.
\end{aligned}
$$

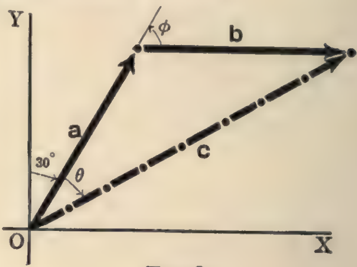


FIG. 8.

Therefore the resultant displacement is about 17.3 cm. along the direction N. 60° E.

PROBLEMS.

1. A vector which points East has a length of 16 cm., and another vector which points Southeast is 25 cm. long. Find the direction and the magnitude of their sum.

2. Find the direction and the magnitude of the difference of the vectors of the last problem.

3. The sum of two vectors is perpendicular to their difference. Show that the vectors are equal in magnitude.

4. The sum and the difference of two vectors have equal magnitudes. Show that the vectors are at right angles to each other.

13. Resolution of Vectors into Components. — The projection of a vector upon a line is called the *component* of the vector along that line. The vectors $\mathbf{a}_x$ and $\mathbf{a}_y$ in Fig. 9, for instance, are the components of $\mathbf{a}$ along the x-axis and the y-axis, respectively. The following relations are evident from the figure and do not need further explanation.

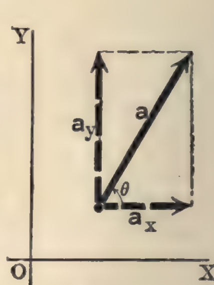


FIG. 9.

* The symbol "$\doteqdot$" will be used to denote approximate equality. Therefore "$\doteqdot$" should be read "equals approximately," or "equals about," or "equals nearly." See table of notations, p. x.

$$\mathbf{a} = \mathbf{a}_x + \mathbf{a}_y, \tag{IV}$$

$$\left.\begin{aligned} a_x &= a \cos \theta, \\ a_y &= a \sin \theta, \end{aligned}\right\} \tag{V}$$

$$a = \sqrt{a_x^2 + a_y^2}, \tag{VI}$$

$$\tan \theta = \frac{a_y}{a_x}. \tag{VII}$$

When **a** has components along all three axes of a rectangular system, Fig. 10, the following equations express the vector in terms of its components.

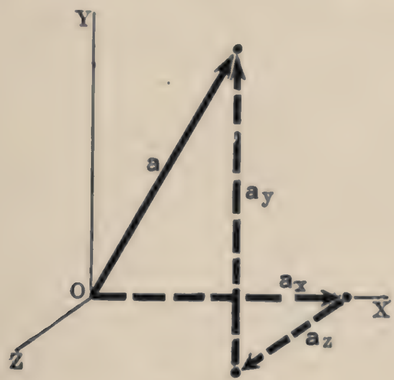


Fig. 10.

$$\mathbf{a} = \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z. \tag{IV'}$$

$$\left.\begin{aligned} a_x &= a \cos \alpha_1, \\ a_y &= a \cos \alpha_2, \\ a_z &= a \cos \alpha_3. \end{aligned}\right\} \tag{V'}$$

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}, \tag{VI'}$$

where α_1, α_2, and α_3 are the angles **a** makes with the coördinate axes.

14. Resultant of Any Number of Vectors. Graphical Methods. —The resultant of a number of vectors **a**, **b**, **c**, etc., may be obtained by either of the following methods.

First: move **b**, without changing either its direction or its magnitude, until its origin falls on the terminus of **a**, then

move **c** until its origin falls on the terminus of **b**, and so on until all the vectors are joined. This gives, in general, an open polygon. Then the resultant is obtained by drawing a vector which closes the polygon and which has its origin at the origin of **a**. The validity of this method will be seen from Fig. 11, where **r** represents the resultant vector. Evidently the resultant vanishes when the given vectors form a closed polygon.

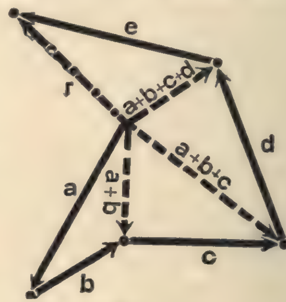


FIG. 11.

Second: draw a system of rectangular coördinate axes; resolve each vector into components along the axes; add the components along each axis geometrically, beginning at the origin. This gives the components of the required vector. Then draw the rectangular parallelopiped determined by these components. The resultant is a vector which has the origin of the axes for its origin and forms a diagonal of the parallelopiped.* This method is based upon the following analytical method.

15. Analytical Method.—Expressing the given vectors and their resultant in terms of their rectangular components, we have

$$\left.\begin{aligned} \mathbf{a} &= \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z, \\ \mathbf{b} &= \mathbf{b}_x + \mathbf{b}_y + \mathbf{b}_z, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \mathbf{r} &= \mathbf{r}_x + \mathbf{r}_y + \mathbf{r}_z. \end{aligned}\right\} \tag{1}$$

Substituting from (1) in the vector equation

$$\mathbf{r} = \mathbf{a} + \mathbf{b} + \mathbf{c} + \cdots \tag{2}$$

and collecting the terms we obtain

$$\begin{aligned} \mathbf{r}_x + \mathbf{r}_y + \mathbf{r}_z = (\mathbf{a}_x + \mathbf{b}_x + \cdots) + (\mathbf{a}_y + \mathbf{b}_y + \cdots) \\ + (\mathbf{a}_z + \mathbf{b}_z + \cdots). \end{aligned} \tag{3}$$

But since the directions of the coördinate axes are indepen-

* When the given vectors are in the same plane the parallelopiped reduces to a rectangle.

dent, the components of $\mathbf{r}$ along any one of the axes must equal the sum of the corresponding components of the given vectors. Therefore (3) can be split into the following three separate equations.

$$\left.\begin{aligned} \mathbf{r}_x &= \mathbf{a}_x + \mathbf{b}_x + \mathbf{c}_x + \cdots, \\ \mathbf{r}_y &= \mathbf{a}_y + \mathbf{b}_y + \mathbf{c}_y + \cdots, \\ \mathbf{r}_z &= \mathbf{a}_z + \mathbf{b}_z + \mathbf{c}_z + \cdots. \end{aligned}\right\} \tag{4}$$

It was shown in § 11 that when two vectors are parallel the algebraic sum of their magnitudes equals the magni-

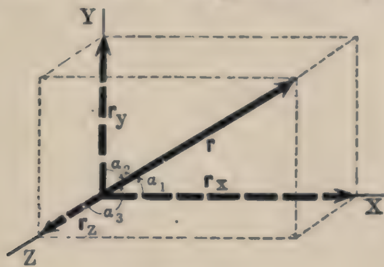


Fig. 12.

tude of their resultant. This result may be extended to any number of parallel vectors. Therefore we can put the vector equations of (4) into the following algebraic forms:

$$\left.\begin{aligned} r_x &= a_x + b_x + c_x + \cdots, \\ r_y &= a_y + b_y + c_y + \cdots, \\ r_z &= a_z + b_z + c_z + \cdots. \end{aligned}\right\} \tag{5}$$

Equations (5) determine $\mathbf{r}$ through the following relations:

$$r = \sqrt{r_x^2 + r_y^2 + r_z^2}, \tag{6}$$

$$\cos \alpha_1 = \frac{r_x}{r}, \quad \cos \alpha_2 = \frac{r_y}{r}, \quad \cos \alpha_3 = \frac{r_z}{r}, \tag{7}$$

where α_1, α_2, and α_3 are the angles $\mathbf{r}$ makes with the axes.

16. Multiplication and Division of a Vector by a Scalar.— When a vector is multiplied or divided by a scalar the result is a vector which has the same direction as the original vector. If, in the equation $\mathbf{b} = m\mathbf{a}$, m be a scalar then $\mathbf{b}$ has the same direction as $\mathbf{a}$ but its magnitude is m times that of $\mathbf{a}$.

ILLUSTRATIVE EXAMPLE.

A man walks 3 miles N. 30° E., then one mile E., then 3 miles S. 45° E., then 4 miles S., then one mile N. 30° W. Find his final position.

Representing the displacements by vectors we obtain the graphical solution given in Fig. 13, where **r** represents the resultant displacement.

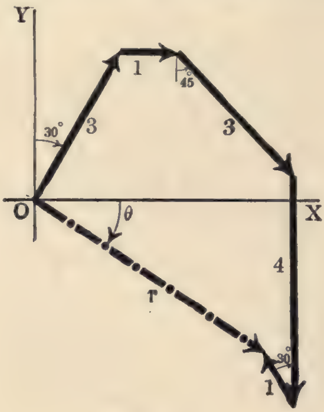


FIG. 13.

In order to find **r** analytically we first determine its components. Thus

$$\begin{aligned} r_x &= [3\cos(60°) + \cos(0°) + 2\cos(-45°) + 4\cos(-90°) \\ &\qquad + \cos(120°)] \text{ miles} \\ &= (2+\sqrt{2}) \text{ miles} \\ &\doteq 3.41 \text{ miles.} \\ r_y &= [3\sin(60°) + \sin(0°) + 2\sin(-45°) + 4\sin(-90°) \\ &\qquad + \sin(120°)] \text{ miles} \\ &= (2\sqrt{3} - \sqrt{2} - 4) \text{ miles} \\ &\doteq -1.95 \text{ miles.} \\ \therefore \quad r &= \sqrt{r_x^2 + r_y^2} \\ &\doteq 3.93 \text{ miles.} \end{aligned}$$

The direction of **r** is given by the following relation.

$$\tan\theta = \frac{r_y}{r_x} \doteq \frac{-1.95}{3.41},$$
$$\therefore \quad \theta \doteq -37°.1.$$

Therefore the final position of the man is about 3.93 miles S. 52°.9 E. from his starting point.

PROBLEMS.

1. The resultant of two vectors which are at right angles to each other is twice the smaller of the two. The magnitude of the smaller vector is a; find the magnitude of the greater vector.

2. In the preceding problem find the resultant vector.

3. Find analytically the sum of three equal vectors which point in the following directions — East, N. 30° W., and S. 30° W.

4. In the preceding problem make use of the first graphical method.

5. In problem 3 make use of the second graphical method.

6. A vector which is 15 cm. long points N. 30° E. Find its components in the following directions.

(a) N. 30° W.	(c) W.	(e) S. 60° E.
(b) N. 60° E.	(d) S. 30° W.	(f) E.

7. A vector $\mathbf{a}$ is in the xy-plane. If 3 is added to a_x and 4 to a_y the direction of the vector is not changed but its magnitude becomes $a_x + a_y$. Find the magnitude and direction of $\mathbf{a}$.

8. Three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ lie in the xy-plane. Find their resultants analytically, taking the magnitudes of their components from the following tables:

	a_x	a_y	b_x	b_y	c_x	c_y
(1)	6,	9,	−5,	2,	0,	10.
(2)	−3,	7,	5,	0,	6,	−8.
(3)	0,	−10,	8,	5,	3,	−2.
(4)	2,	0,	−6,	4,	0,	8.

9. In the preceding problem make use of the second graphical method.

10. Straight horizontal tunnels in a mine connect the points P_1, P_2, P_3, and P_4, in the given order. The length of each tunnel and the angle it makes with the meridian are given in the following tables. Find the lengths and directions of the tunnels which have to be dug in order to connect P_1 with P_3 and P_4 directly.

$P_1P_2 = 200$ feet, and makes 30° with the meridian.
$P_2P_3 = 100$ feet, and makes 120° with the meridian.
$P_3P_4 = 400$ feet, and makes 300° with the meridian.

11. Work out the preceding problem by the first graphical method.

12. Work out problem 10 by the second graphical method.

13. Find the direction and magnitude of the force experienced by an electrical charge of five units placed at one vertex of an equilateral triangle due to two unlike charges of 10 units each placed at the other vertices. The sides of the triangle are 2 cm.

CHAPTER II.

EQUILIBRIUM OF A PARTICLE.

ACTION AND REACTION. FORCE.

17. Particle. — A body whose dimensions are negligible is called a *particle*. In a problem any body may be considered as a particle so long as it does not tend to rotate. Even when the body rotates it may be considered as a particle if its rotation does not enter into the problem. For instance, in discussing the motion of the earth in its orbit the earth is considered as a particle, because its rotation about its axis does not enter into the discussion.

18. Degrees of Freedom.—The number of independent ways in which a body can move is called the number of *degrees of freedom* of its motion. It equals the number of coördinates which are necessary in order to specify completely the position of the body. A free particle can move in three independent directions, that is, along the three axes of a system of rectangular coördinates, therefore it has three degrees of freedom. When the particle is constrained to move in a plane its freedom is reduced to two degrees, because it can move only in two independent directions. When it is constrained to move in a straight line it has only one degree of freedom.

19. Force.—While considering the motion or the equilibrium of a body our attention is claimed not only by that body but also by others which act upon it. In order to insure concentration of attention problems in Dynamics are simplified in the following manner. All bodies are eliminated, except the one the motion of which is being discussed, and their actions upon the latter are represented by certain vector magnitudes known as *forces*. As an illustration consider

the equilibrium of the shaded part of the rope in Fig. 14a. The shaded part is acted upon by the adjoining sections of the rope. Therefore we consider the shaded part alone and represent the actions of the adjoining parts by the forces $-\mathbf{F}$ and $\mathbf{F}$, as shown in Fig. 14b.

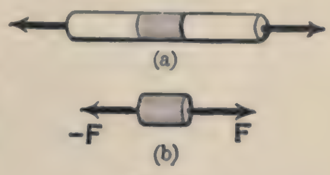


FIG. 14.

20. Definition of Force.—*Force is a vector magnitude which represents the action of one body upon another.* The interaction between two bodies takes place across an area, while the forces which represent them are supposed to be applied at one point. Therefore the introduction of the idea of force presupposes the simplification of dynamical problems which is obtained by considering bodies as single particles, or as a system of particles.

21. Internal Force.—A force which represents the action of one part of a body upon another part of the same body is called an internal force.

22. External Force.—A force which represents the action of one body upon another body is called an external force.

23. Unit Force.—The engineering unit of force among English speaking people is the *pound*. The pound is the weight, in London, of a certain piece of platinum kept by the British government.

24. The Law of Action and Reaction.—The fundamental law of Mechanics is known as the *law of action and reaction*. Newton (1692–1727), who was the first to formulate it, put the law in the following form.

"*To every action there is an equal and opposite reaction, or the mutual actions of two bodies are equal and oppositely directed.*"

Let us apply this law to the interaction between a book and the hand in which you hold it. Your hand presses upward upon the book in order to keep it from falling,

while the book presses downward upon your hand. The law states that the action of your hand equals the reaction of the book and is in the opposite direction. The book reacts upon your hand because the earth attracts it. When your hand and the earth are the only bodies which act upon the book, the action of your hand equals and is opposite to the action of the earth. In other words the sum of the two actions is nil. Generalizing from this simple illustration we can put the law into the following form:

To every action there is an equal and opposite reaction, or the sum of all the actions to which a body or a part of a body is subject at any instant vanishes:

$$\Sigma \mathbf{A} = 0.^* \tag{A}$$

25. Condition for the Equilibrium of a Particle. — The condition of equilibrium of a particle is obtained by replacing the term "action" by the term "force" in the last form of the fundamental law and then stating it in the form of a condition. Thus — *in order that a particle be in equilibrium the sum of all the forces which act upon it must vanish.*

In other words if $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, . . . , $\mathbf{F}_n$ are the forces which act upon a particle, then the vector equation

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots + \mathbf{F}_n = 0 \tag{I}$$

must be satisfied in order that the particle be in equilibrium. Equation (I) is equivalent to stating that when the forces are added graphically they form a closed polygon. But when the sum of a number of vectors vanishes the sum of their components also vanishes. Therefore we must have

$$\left.\begin{aligned} \mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_n = 0, \\ \mathbf{Y}_1 + \mathbf{Y}_2 + \cdots + \mathbf{Y}_n = 0, \\ \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_n = 0, \end{aligned}\right\} \tag{II$'$}$$

where $\mathbf{X}_i$, $\mathbf{Y}_i$, and $\mathbf{Z}_i$ are the components of $\mathbf{F}_i$.* Since the vectors in each of the equations of (II′) are parallel we can

* See table of notations.

write them as algebraic equations. Therefore we have the following equations for the analytical form of the condition of equilibrium of a particle.

$$\left.\begin{aligned}\Sigma X &\equiv X_1 + X_2 + \cdots + X_n = 0,^*\\ \Sigma Y &\equiv Y_1 + Y_2 + \cdots + Y_n = 0,\\ \Sigma Z &\equiv Z_1 + Z_2 + \cdots + Z_n = 0.\end{aligned}\right\} \qquad \text{(II)}$$

The condition of equilibrium may, therefore, be stated in the following form.

In order that a particle be in equilibrium the algebraic sum of the components of the forces along each of the axes of a rectangular system of coördinates must vanish.

The following rules will be helpful in working out problems on the equilibrium of a particle.

First. Represent the particle by a point and the action of each body which acts upon it by a properly chosen force-vector. Be sure that all the bodies which act upon the particle are thus represented.

Second. Set the sums of the components of the forces along properly chosen axes equal to zero.

Third. If there are not enough equations to determine the unknown quantities, obtain others from the geometrical connections of the problem.

Fourth. Solve these equations for the required quantities.

Fifth. Discuss the results.

ILLUSTRATIVE EXAMPLES.

1. A particle suspended by a string is pulled aside by a horizontal force until the string makes an angle α with the vertical. Find the tensile force in the string and the magnitude of the horizontal force in terms of the weight of the particle.

The particle is acted upon by three bodies, namely, the earth, the string, and the body which exerts the horizontal force. Therefore, we

* The relation $\Sigma X \equiv X_1 + X_2 + \cdots + X_n$ is not an equation. It merely states that ΣX is identical with and is an abbreviation for $X_1 + X_2 + \cdots + X_n$.

represent the actions of these bodies by three force-vectors, **W**, **T**, and **F**, Fig. 15, and then apply the conditions of equilibrium. Setting equal to zero the sums of the components of the forces along the x- and y-axes, we get

$$\Sigma X \equiv F - T \sin \alpha = 0. \qquad \text{(a)}$$
$$\Sigma Y \equiv -W + T \cos \alpha = 0. \qquad \text{(b)}$$

Solving equations (a) and (b) we have

$$T = \frac{W}{\cos \alpha},$$

and

$$F = T \sin \alpha$$
$$= W \tan \alpha.$$

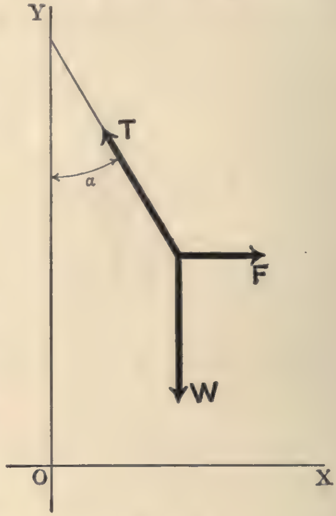


Fig. 15.

Discussion. — When $\alpha = 0$, $T = W$ and $F = 0$. When $\alpha = \frac{\pi}{2}$, $T = \infty$ and $F = \infty$. Therefore no finite horizontal force can make the string perfectly horizontal.

2. A uniform bar, of weight W and length a, is suspended in a horizontal position by two strings of equal length l. The lower ends of the strings are fastened to the ends of the bar and the upper ends to a peg. Find the tensile force in the strings.

The bar is acted upon by three bodies, namely the earth and the two strings. We represent their actions by the forces **W**, $\mathbf{T}_1$, and $\mathbf{T}_2$, Fig. 16a. The tensile forces of the strings act at the ends of the bar. On the other hand the weight is distributed all along the rod. But we may consider it as acting at the middle point, as in Fig. 16a, or we may replace the rod by two particles of weight $\frac{W}{2}$ each, as shown in Fig. 16b. In the last case the rigidity of the bar which prevents its ends from coming together is represented by the forces **F** and $-$**F**.

Considering each particle separately and setting equal to zero the sums of the components of the forces along the axes, we obtain

$$\Sigma X \equiv T_1 \cos \alpha - F = 0,$$
$$\Sigma Y \equiv T_1 \sin \alpha - \frac{W}{2} = 0,$$

for the first particle, and

$$\Sigma X \equiv -T_2 \cos \alpha + F = 0,$$
$$\Sigma Y \equiv T_2 \sin \alpha - \frac{W}{2} = 0,$$

for the second particle. It follows from these equations that

$$T_1 = T_2$$
$$= \frac{W}{2 \sin \alpha}$$
$$= \frac{l}{\sqrt{4 l^2 - a^2}} W.$$

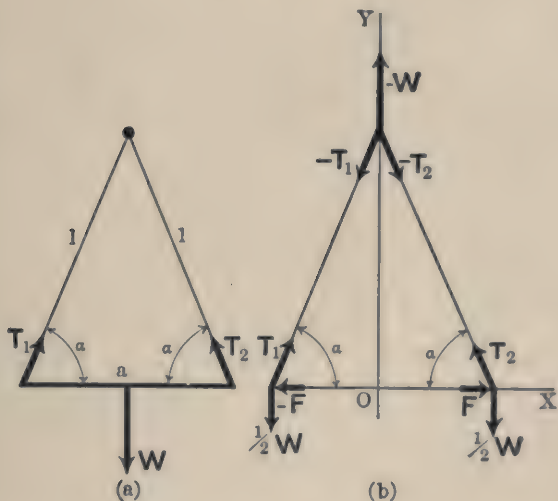


FIG. 16.

DISCUSSION. — The tensile force of the strings increases indefinitely as their total length approaches that of the bar. On the other hand as the length of the strings becomes very large compared with that of the bar the tensile force approaches $\frac{W}{2}$ as a limit.

The problem can be solved also by considering the forces acting on the peg, as shown in Fig. 16b.

PROBLEMS.

1. Show that when a particle is in equilibrium under the action of two forces the forces must lie in the same straight line.

2. Show that when a particle is in equilibrium under the action of three forces the forces lie in the same plane.

3. Find the horizontal force which will keep in equilibrium a weight of 150 pounds on a smooth inclined plane which makes 60° with the horizon.

4. A ring of weight W is suspended by means of a string of length l, the ends of which are attached to two points on the same horizontal line. Find the tensile force of the string if the distance between its ends is d. Also discuss the limiting cases in which l approaches d or becomes very large compared with it.

5. A body of weight W is suspended by two strings of lengths l_1 and l_2. The upper end of each string is attached to a fixed point in the same horizontal line. Find the tensile forces in the strings if the distance between the two points is d.

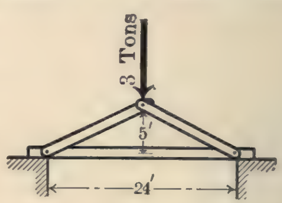
6. A weight is suspended by four equal strings, the upper ends of which are attached to the vertices of a horizontal square. Find the tensile forces in the strings.

7. A particle is in equilibrium on a smooth inclined plane under the action of two equal forces, the one acting along the plane upwards and the other horizontally. Find the inclination of the plane.

8. Apply the conditions of equilibrium to find the magnitude and direction of the resultant of a number of forces acting upon a particle.

9. Two spheres of equal radius and equal weight are in equilibrium in a smooth hemispherical bowl; find the reactions between the two spheres and between the spheres and the bowl.

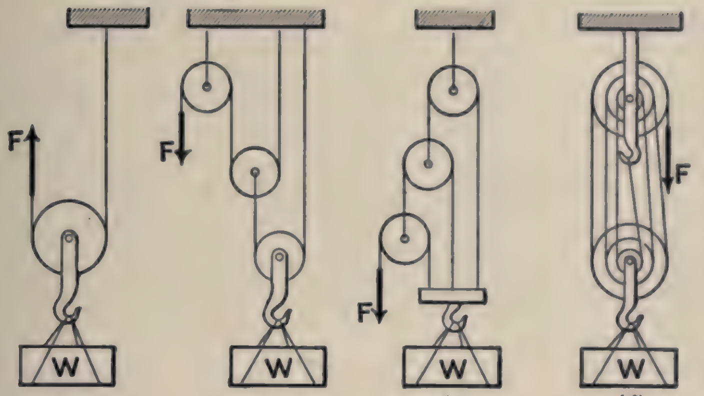
10. The ends of a string, 60 cm. long, are fastened to two points in the same horizontal line and at a distance of 40 cm. apart; two weights are hung from points in the string 25 cm. and 20 cm. from the ends. Find the ratio of the weights if the part of the string between them is horizontal.



11. A single triangular truss of 24 feet span and 5 feet depth supports a load of 3 tons at the apex. Find the forces acting on the rafters and the tie rod.

12. A particle of weight W can be kept in equilibrium upon a smooth inclined plane by a force F_1 acting horizontally; it can also be kept in equilibrium by a force F_2 acting parallel to the plane. Express W in terms of F_1 and F_2.

13. In the following arrangements of pulleys find the relation between **F** and **W**.



SLIDING FRICTION.

26. Frictional Force.—Consider the forces acting upon a body which is in equilibrium on a rough inclined plane, Fig. 17. The body is acted upon by two forces, namely, its weight, W, and the reaction of the plane, R. The reaction of the plane is the result of two distinct and independent forces. One of these, N, is perpendicular to the plane and is called the *normal reaction.* The other, F, is along the plane and is called the *frictional force.* The normal reaction is due to the rigidity of the plane. It resists the tendency of the body to go through the plane. The frictional force is due to the roughness of the contact between the body and the plane. It prevents the body from sliding down the plane.

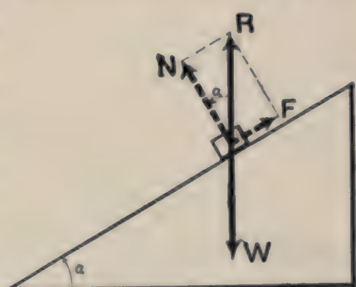


Fig. 17.

27. Angle of Friction. — As we increase the angle of elevation of the inclined plane a certain definite angle will be reached

when the equilibrium is disturbed and the body begins to slide down the plane. This angle is called the *angle of friction.* This definition for the angle of friction does not hold when the body is acted upon by other forces besides its weight and the reaction of the plane. The following definition, however, is valid under all circumstances: *The angle of friction equals the angle which the total reaction makes with the normal to the surface of contact when the body is on the point of motion.*

28. Coefficient of Friction.—Denoting the angle of friction by ϕ, we obtain

$$F = R \sin \phi,$$

$$N = R \cos \phi.$$

Therefore

$$F = N \tan \phi$$

$$= \mu N, \tag{III}$$

where $\mu = \tan \phi$ and is called the *coefficient of friction.* The angle of friction and consequently the coefficient of friction are constants which depend upon the surfaces in contact. *The last four equations hold true only when the body is on the point of motion.*

29. Static and Kinetic Friction.—The friction which comes into play is called static friction if the body is at rest and kinetic friction if it is in motion.

30. Laws of Friction.—The following statements, which are generalizations derived from experimental results, bring out the important properties of friction. They hold true within certain limits and are only approximately true even within these limits.

1. Frictional forces come into play only when a body is urged to move.

2. Frictional forces always act in a direction opposite to that in which the body is urged to move.

3. Frictional force is proportional to the normal reaction, $F = \mu N$.

4. Frictional force is independent of the area of contact.

5. The static frictional force which comes into play is not greater than that which is necessary to keep the body in equilibrium.

6. Kinetic friction is smaller than static friction.

Laws 1 to 4 hold true for both static and kinetic friction. The coefficient of friction between two bodies depends upon the condition of surfaces in contact. Therefore the value of μ is not a perfectly definite constant for a given pair of substances in contact.

The values given in the following table are averages of values obtained by several experimenters.

Materials in contact.	Condition of surfaces in contact.	Coefficient of friction.	
		Static.	Kinetic.
Wood on wood	Dry	.50	.36
Wood on wood	Wet	.68	.25
Wood on wood	Polished and greased	.35	.12
Heavy rope on wood	Dry	.60	.40
Heavy rope on wood	Wet	.80	.35
Cast iron on cast iron	Dry	.24	.18
Cast iron on cast iron	Greased	.15	.13
Cast iron on oak	Wet	.65	—
Leather on cast iron	——	.30	—

ILLUSTRATIVE EXAMPLES.

1. A body which is on a rough horizontal floor can be brought to the point of motion by a force which makes an angle α with the floor. Find the reaction of the floor and the coefficient of friction.

The body is acted upon by three forces, Fig. 18,

P, the given force,
W, the weight of the body,
R, the reaction of the floor.

Replacing **R** by its components **F** and **N**, and applying the conditions of equilibrium, we obtain

$$\Sigma X = P \cos \alpha - F = 0,$$
$$\Sigma Y = P \sin \alpha + N - W = 0.$$

Therefore

$$F = P \cos \alpha,$$
$$N = W - P \sin \alpha,$$

and
$$R = \sqrt{F^2 + N^2}$$
$$= \sqrt{P^2 + W^2 - 2\,PW \sin \alpha}.$$

But since the body is on the point of motion the relation $F = \mu N$ holds. Therefore
$$\mu = \frac{F}{N} = \frac{P \cos \alpha}{W - P \sin \alpha}.$$

DISCUSSION. — (a) When $\alpha = 0$, $R = \sqrt{P^2 + W^2}$ and $\mu = \frac{P}{W}$. (b) When $\alpha = \frac{\pi}{2}$, $R = P - W = 0$, therefore $P = W$, and μ is indeterminate. (c) When $P = 0$, $\mu = 0$, and $R = W$.

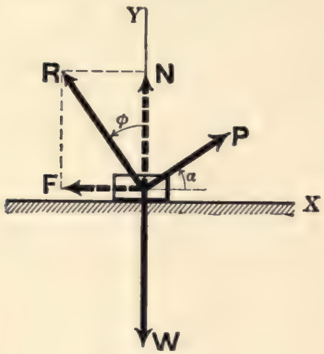


FIG. 18.

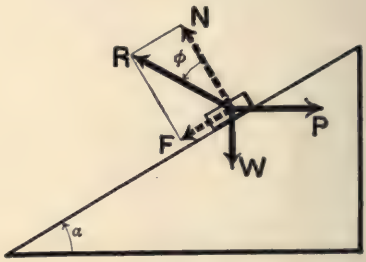


FIG. 19.

2. A body which rests upon a rough inclined plane is brought to the point of motion up the inclined plane by a horizontal force. Find μ and R.

The body is acted upon by three forces, Fig. 19,

P, the horizontal force,
W, the weight,
R, the reaction of the plane.

Replacing **R** by its components **F** and **N**, and taking the axes along and at right angles to the plane, we obtain
$$\Sigma X \equiv P \cos \alpha - F - W \sin \alpha = 0,$$
$$\Sigma Y \equiv -P \sin \alpha + N - W \cos \alpha = 0.$$

Therefore
$$F = P \cos \alpha - W \sin \alpha,$$
$$N = P \sin \alpha + W \cos \alpha,$$
$$R = \sqrt{F^2 + N^2}$$
$$= \sqrt{P^2 + W^2},$$

and
$$\mu = \frac{F}{N} = \frac{P\cos\alpha - W\sin\alpha}{P\sin\alpha + W\cos\alpha}.$$

DISCUSSION. — (a) When $\alpha = 0$, $\mu = \frac{P}{W}$, and $R = W\sqrt{\mu^2 + 1}$. (b) When $P = 0$, $\mu = -\tan\alpha$; therefore $\alpha = -\phi$, that is, the inclined plane must be tipped in the opposite direction and must be given an angle of elevation equal to the angle of friction in order that motion may take place towards the positive direction of the x-axis.

PROBLEMS.

1. A body which weighs 100 pounds is barely started to move on a rough horizontal plane by a force of 150 pounds acting in a direction making 30° with the horizon. Find R and μ.

2. A body placed on a rough inclined plane barely starts to move when acted upon by a force equal to the weight of the body. Find the coefficient of friction, (a) when the force is normal to the plane; (b) when it is parallel to the plane.

3. A horizontal force equal to the weight of the body has to be applied in order to just start a body into motion on a horizontal floor. Find the coefficient of friction.

4. A weight W rests on a rough inclined plane, which makes an angle α with the horizon. Find the smallest force which will move the weight if the coefficient of friction is μ.

5. How would you determine experimentally the coefficient of friction between two bodies?

6. A weight of 75 pounds rests on a rough horizontal floor. Find the magnitude of the least horizontal force which will move the body if the coefficient of friction is 0.4; also find the reaction of the plane.

7. A particle of weight $\mathbf{W}$ is in equilibrium on an inclined plane under the action of a force $\mathbf{F}$, which makes the magnitude of the normal pressure equal W. The coefficient of friction is μ and the angle of elevation of the inclined plane is α. Find the magnitude and direction of the force.

8. An insect starts from the highest point of a sphere and crawls down. Where will it begin to slide if the coefficient of friction between the insect and the sphere is $\frac{1}{3}$?

9. The greatest force, which can keep a particle at rest, acting along an inclined plane, equals twice the least force. Find the coefficient of friction. The angle of elevation of the plane is α.

31. Resultant of a System of Forces. — The resultant of a number of forces which act upon a particle is a force which

is equivalent to the given forces. There are two criteria by which this equivalence may be tested. First: The resultant force will give the particle the same motion, when applied to it, as that imparted by the given system of forces. We cannot use this test just now because we have not yet studied motion. Second: When the resultant force is reversed and applied to the particle simultaneously with the given forces the particle remains in equilibrium.

According to the second criterion, therefore, the resultant, $\mathbf{R}$, of the forces $\mathbf{F}_1$, $\mathbf{F}_2$, . . . , $\mathbf{F}_n$, must satisfy the equation

$$\left.\begin{array}{rl} & -\mathbf{R}+(\mathbf{F}_1+\mathbf{F}_2+\cdots+\mathbf{F}_n)=0, \\ \text{or} & \mathbf{R}=\mathbf{F}_1+\mathbf{F}_2+\cdots+\mathbf{F}_n. \end{array}\right\} \qquad \text{(IV}'\text{)}$$

Splitting the last equation into three algebraic equations, we obtain

$$\left.\begin{array}{l} X=X_1+X_2+\cdots+X_n, \\ Y=Y_1+Y_2+\cdots+Y_n, \\ Z=Z_1+Z_2+\cdots+Z_n, \end{array}\right\} \qquad \text{(IV)}$$

where $\mathbf{X}_i$, $\mathbf{Y}_i$, and $\mathbf{Z}_i$ are the components of $\mathbf{F}_i$.

The magnitude of $\mathbf{R}$ is given by the relation

$$R=\sqrt{X^2+Y^2+Z^2}, \qquad \text{(V}'\text{)}$$

while the direction is obtained from the following expressions for its direction cosines.

$$\cos\alpha_1=\frac{X}{R}, \quad \cos\alpha_2=\frac{Y}{R}, \quad \cos\alpha_3=\frac{Z}{R}. \qquad \text{(VI}'\text{)}$$

Special Case.—When the forces lie in the xy-plane the z-component of each force equals zero. Therefore we have

$$R=\sqrt{X^2+Y^2}, \qquad \text{(V)}$$

and

$$\tan\theta=\frac{Y}{X}, \qquad \text{(VI)}$$

where θ is the angle $\mathbf{R}$ makes with the x-axis.

PROBLEMS.

1. Three men pull on a ring. The first man pulls with a force of 50 pounds toward the N. 30° W. The second man pulls toward the S. 45° E. with a force of 75 pounds, and the third man pulls with a force of 100 pounds toward the west. Determine the magnitude and direction of the resultant force.

2. Show that the resultant of two forces acting upon a particle lies in the plane of the given forces.

3. Show that the line of action of the resultant of two forces lies within the angle made by the forces.

4. Find the direction and magnitude of the resultant of three equal forces which act along the axes of a rectangular system of çoördinates.

GENERAL PROBLEMS.

1. A particle is in equilibrium under the action of the forces $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$. Prove that

$$\frac{P}{\sin(\mathbf{Q}, \mathbf{R})} = \frac{Q}{\sin(\mathbf{P}, \mathbf{R})} = \frac{R}{\sin(\mathbf{P}, \mathbf{Q})},$$

where $(\mathbf{Q}, \mathbf{R})$, etc., denote the angles between $\mathbf{Q}$ and $\mathbf{R}$, etc.

2. Two particles of weights W_1 and W_2 rest upon a smooth sphere of radius a. The particles are attached to the ends of a string of length l, which passes over a smooth peg vertically above the center of the sphere. If h is the distance between the peg and the center of the sphere, find (1) the position of equilibrium of the particles, (2) the tensile force in the string, and (3) the reaction of the sphere.

3. The lengths of the mast and the boom of a derrick are a and b respectively. Supposing the hinges at the lower end of the boom and the pulley at the upper end to be smooth, find the angle the boom makes with the vertical when a weight W is suspended in equilibrium.

4. Find the tensile force in the chain and the compression in the boom of the preceding problem.

5. Two rings of weights W_1 and W_2 are held on a smooth circular wire in a vertical plane by means of a string subtending an angle 2α at the center. Show that the inclination of the string to the horizon is given by

$$\tan\theta = \frac{W_1 - W_2}{W_1 + W_2}\tan\alpha.$$

6. A bridge, Fig. (a), of 60-foot span and 40-foot width has two queen-post trusses 9 feet deep. Each truss is divided into three equal parts by two

posts. What are the stresses in the different parts of the trusses when there is a load of 150 pounds per square foot of floor space?

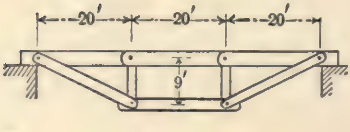


FIG. (a).

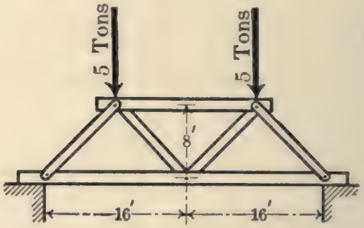


FIG. (b).

7. Find the force in one of the members of the truss of figure (b).

8. A weight rests upon a smooth inclined plane, supported by two equal strings the upper ends of which are fastened to two points of the plane in the same horizontal line. Find the tensile force in the strings and the reaction of the plane.

9. In the preceding problem suppose the plane to be rough.

10. A particle is suspended by a string which passes through a smooth ring fastened to the highest point of a circular wire in a vertical plane. The other end of the string is attached to a smooth bead which is movable on the wire. Find the position of equilibrium supposing the bead and the suspended body to have equal weights.

11. A particle is in equilibrium on a rough inclined plane under the action of a force which acts along the plane. If the least magnitude of the force when the inclination of the plane is α equals the greatest magnitude when it is α_2, show that $\phi = \dfrac{\alpha_1 - \alpha_2}{2}$, where ϕ is the angle of friction.

12. Two weights W_1 and W_2 rest upon a rough inclined plane, connected by a string which passes through a smooth pulley in the plane. Find the greatest inclination the plane can be given without disturbing the equilibrium.

13. Two equal weights, which are connected by a string, rest upon a rough inclined plane. If the direction of the string is along the steepest slope of the plane and if the coefficients of friction are μ_1 and μ_2, find the greatest inclination the plane can be given without disturbing the equilibrium.

14. In the preceding problem find the tensile force in the string.

15. One end of a uniform rod rests upon a rough peg, while the other end is connected, by means of a string, to a point in the horizontal plane which contains the peg. When the rod is just on the point of motion it

is perpendicular to the string. Show that $2l = \mu a$, where l is the length of the string, a that of the rod, and μ the coefficient of friction.

16. A particle resting upon an inclined plane is at the point of motion under the action of the force $\mathbf{F}$, which acts downward along the plane. If the angle of elevation of the plane is changed from α_1 to α_2 and the direction of the force reversed the particle will barely start to move up the plane. Express μ in terms of α_1 and α_2.

17. A string, which passes over the vertex of a rough double inclined plane, supports two weights. Show that the plane must be tilted through an angle equal to twice the angle of friction, in order to bring it from the position at which the particles will begin to move in one direction to the position at which they will begin to move in the opposite direction.

18. Three equal spheres are placed on a smooth horizontal plane and are kept together by a string, which surrounds them in the plane of their centers. If a fourth equal sphere is placed on top of these, prove that the tensile force in the string is $\frac{W}{3\sqrt{6}}$, where W is the weight of each sphere.

19. Three equal hemispheres rest with their bases upon a rough horizontal plane and are in contact with one another. What is the least value of μ which will enable them to support a smooth sphere of the same radius and material?

20. If the center of gravity of a rod is at a distance a from one end and b from the other, find the least value of μ which will allow it to rest in all positions upon a rough horizontal ground and against a rough vertical wall.

21. A string, which is slung over two smooth pegs at the same level, supports two bodies of equal weight W at the ends, and a weight W at the middle by means of a smooth ring through which it passes. Find the position of equilibrium of the middle weight.

CHAPTER III.

EQUILIBRIUM OF RIGID BODIES.

TRANSLATION AND ROTATION.

32. Rigid Body. — There are problems in which bodies cannot be treated as single particles. In such cases they are considered to be made up of a great number of discrete particles. A body is said to be *rigid* if the distances between its particles remain unchanged whatever the forces to which it may be subjected. There are no bodies which are strictly rigid. All bodies are deformed more or less under the action of forces. But in most problems discussed in this book ordinary solids may be treated as rigid bodies.

33. Motion of a Rigid Body. — A rigid body may have two distinct types of motion. When the body moves so that its particles describe straight paths it is said to have a *motion of translation*. Evidently the paths of the particles are parallel, Fig. 20. If the particles of the body describe circular paths it is said to have a *motion of rotation*. The planes of the circles are parallel, while their centers lie on a straight line perpendicular to these planes, which is called the *axis of rotation*. The motion of a flywheel is a well-known example of motion of rotation. Suppose A, Fig. 21, to be a rigid body which is brought from the position A to the position A' by a motion of rotation about an axis through the point O perpendicular to the plane of the paper, then the paths of its particles

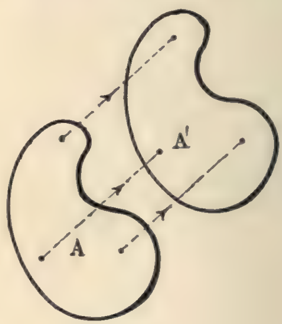


FIG. 20.

are arcs of circles whose planes are parallel to the plane of the paper and whose centers lie on the axis of rotation.

Fig. 21.

34. Uniplanar Motion. — When a rigid body moves so that each of its particles remains at a constant distance from a fixed plane the motion is said to be *uniplanar*. The fixed plane is called the *guide plane*.

35. Theorem I. — *Uniplanar motion of a rigid body consists of a succession of infinitesimal rotational displacements.*

Suppose the rigid body A, Fig. 22, to describe a uniplanar motion parallel to the plane of the paper and let A and A' be any two positions occupied by the body. Then it may be brought from A to A' by a rotational displacement about an axis the position of which may be found in the following manner. Let P and Q be the positions of any two particles of the body in a plane parallel to the plane of the paper when the body is at the position A, and P' and Q' be the positions of the same particles when the body occupies the position A'. Then the desired axis is perpendicular to the plane of the paper and passes through the point of intersection of the perpendicular bisectors of the lines PP' and QQ', drawn in the plane determined by these lines.

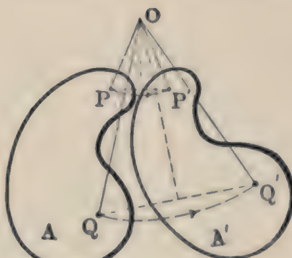


Fig. 22.

Therefore the body can be brought from any position A to any other position A' by a single rotational displacement. The actual motion between A and A' will be, in general,

quite different from the simple rotation by which we accomplished the passage of the body from one of these positions to the other. But the result, which we have just obtained, is true not only for positions which are separated by finite distances but also for positions which are infinitely near each other. Therefore by giving the body infinitesimal rotational displacements about properly chosen axes it may be made to assume all the positions which it occupies during its actual motion.

36. Instantaneous Axis. — As the body is made to occupy the various positions of its actual motion the axis of rotation moves at right angles to itself and generates a cylinder whose elements are perpendicular to the guide plane. The elements of the cylinder are called *instantaneous axes*, because each acts as the axis of rotation at the instant when the body occupies a certain position. The curve of intersection of the cylinder and the guide plane is called the *centrode.*

The motion of a cylinder which rolls in a larger cylinder is a simple example of uniplanar motion. In this case the common element of contact is the instantaneous axis. As the cylinder rolls different elements of the fixed cylinder become the axis of rotation.

Motion of translation and motion of rotation are special cases of uniplanar motion. In motion of translation the axis of rotation is infinitely far from the moving body. In rotation the cylinder formed by the instantaneous axes reduces to a single line, i.e., the axis of rotation.

37. Theorem II. — *Rotation about any axis is equivalent to a rotation through the same angle about a parallel axis and a translation in a direction perpendicular to it.*

The truth of this theorem will be seen from Fig. 23, where the rigid body A is brought from the position A to the position A' by a single rotation about an axis through the point O perpendicular to the plane of the paper. This displace-

ment may be produced also by rotating the body to the position A'' and then translating it to the position A'.

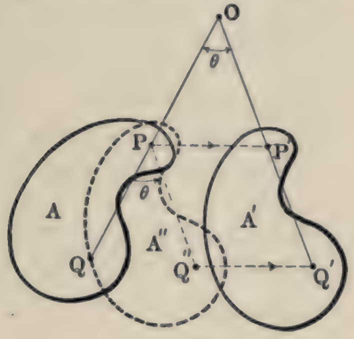


Fig. 23.

PROBLEMS.

1. Show that in theorem II the order of the rotation and of the translation may be changed.

2. Show that the converse of theorem II is true.

38. Theorem III.—*The most general displacement of a rigid body can be obtained by a single translation and a single rotation.*

Let A and A' be any two positions occupied by the rigid body and P and P' be the corresponding positions of any one

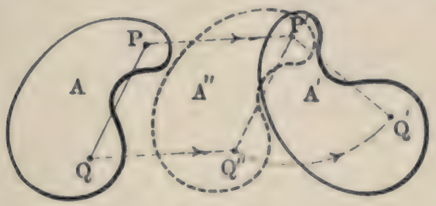


Fig. 24.

of its particles. Then the body may be brought from A to A' by giving it a motion of translation which will bring the particle from P to P' and then rotating the body about a properly chosen axis through P'. A special case of this

theorem is illustrated in Fig. 24, where the direction of the translation is perpendicular to the axis of rotation.

39. Theorem IV.—*The most general displacement of a rigid body can be obtained by a displacement similar to that of a screw in its nut, that is, by a rotation about an axis and a translation along it.*

This theorem states that the axis of rotation of the last theorem can be so chosen that the translation is along the axis of rotation. In theorem III let PP', Fig. 25, be the path of any point of the body described during the translation and BB be the line about which the body is rotated. Draw CC through P parallel to BB and drop the perpendicular $P'P''$ upon CC. The displacement may be accomplished now in the following three stages. First: translate the body along the line CC until the point which was at P arrives at P''. Second: translate the body along $P''P'$ until the point arrives at P'. Third: rotate the body about BB until it comes to the desired position. But by theorem II the last two operations can be accomplished by a single rotation about CC. Therefore the desired displacement can be obtained by a translation along and a rotation about the line CC.

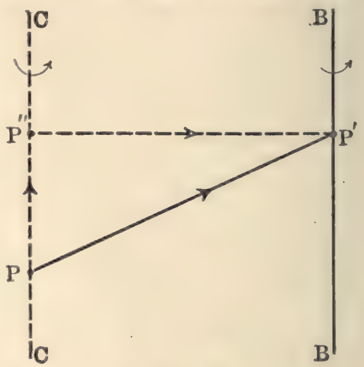


Fig. 25.

Evidently the last theorem holds for infinitesimal displacements as well as for finite displacements; therefore however complicated the motion of a rigid body it can be reproduced by a succession of infinitesimal *screw-displacements*, each displacement taking the body from one position which it has occupied during the motion to another position infinitely near it. Thus at every instant of its motion the rigid body is displaced like a screw in its nut. In general

the pitch and the direction of the axis of the screw-motion change from instant to instant. In the case of the motion of a screw in its nut these do not change.

Translation and rotation are special cases of screw-motion. When the pitch of a screw is made smaller and smaller it advances less and less during each revolution. Therefore if the pitch is made to vanish the screw does not advance at all when it is rotated. Thus rotation is a special case of screw-motion in which the pitch is zero. On the other hand as the pitch of the screw is made greater and greater the screw advances more and more during each revolution. Therefore at the limit when the pitch is infinitely great the motion of the screw becomes a motion of translation. Thus translation is a special case of screw-motion in which the pitch is infinitely great.

LINEAR AND ANGULAR ACTION. TORQUE.

40. Two Types of Action.—We have seen that a rigid body may have two different and independent types of motion, namely, motion of translation and motion of rotation. These motions are the results of two independent and entirely different kinds of actions to which a rigid body is capable of being subjected. We will differentiate between these two types of action by adding the adjectives "linear" and "angular" to the term "action." Thus the action which tends to produce translation will be called *linear action* and that which tends to produce rotation, *angular action*.

41. Torque.—The vector magnitude which represents the angular action of one body upon another is called *torque*.

42. Couple.—Although a single force is not capable of giving a rigid body a motion of pure rotation, two or more external forces will do it when properly applied. The simplest system of forces which is capable of producing rotation is known as a *couple*. It consists of two equal and opposite forces which are not in the same line, Fig. 26.

It is evident from Fig. 26 that a couple is capable of giving a rigid body a motion of rotation. But this is not enough to show that the effect produced by a couple is the same as that produced by a torque. We must show also that the couple is not capable of producing a motion of translation. Consider the rigid body A, Fig. 27, which is acted upon by a couple. Suppose the couple did tend to

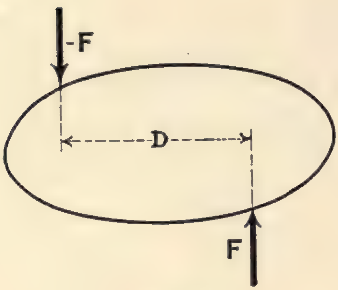


Fig. 26.

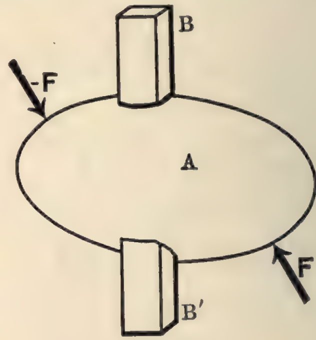


Fig. 27.

produce a translation in a direction BB'. Then pass through the body a smooth bar of rectangular cross-section in the direction of the supposed motion, so that the body is free to move along the bar but not free to rotate. When this constraint is imposed upon the rigid body it behaves like a particle and therefore cannot be given a motion by two equal and opposite forces. But since any motion in the direction BB' is not affected by the presence of the bar, the assumption that the couple produces a motion of translation along BB' must be wrong. Hence we see that when the bar is taken out the motion due to the couple will be one of pure rotation.

43. Measure of Torque. — When a rigid body is in equilibrium under the action of two couples it is always found that the product of one of the forces of one couple by the distance apart of the forces of the same couple equals the

corresponding product for the other couple. In order, for instance, that the rigid body A, Fig. 28, be in equilibrium, we must have

$$FD = F'D'.$$

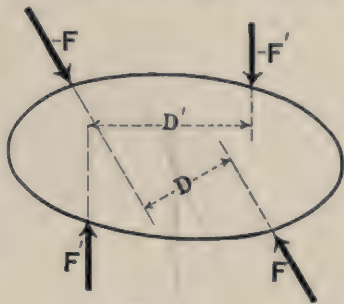


FIG. 28.

Therefore the product FD is the measure of the torque of the couple formed by the forces **F** and **−F**, the lines of action of which are separated by the distance D. Thus denoting the torque of a couple by G, we have

$$G = FD. \tag{I}$$

The distance D is called the *arm* of the couple and the plane of the forces the *plane* of the couple.

44. Unit Torque. — The torque of a couple whose forces are one pound each and whose arm is one foot is the unit of torque. The symbol for the unit torque is the lb. ft.

45. Vector Representation of Torque. — Torque is a vector magnitude and is represented by a vector which is perpen-

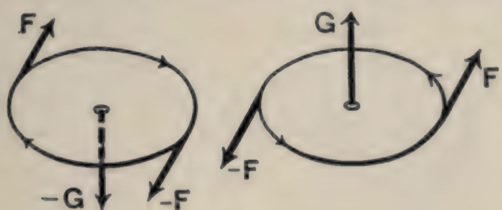


FIG. 29.

dicular to the plane of the couple. The vector points away from the observer when the couple tends to rotate the body in the clockwise direction and points towards the observer when it tends to rotate the body in the counterclockwise direction, Fig. 29. In the first case the torque is considered to be negative and in the second case positive.

46. Equal Couples. — Two couples are equal when the vectors which represent their torques are equal in magnitude and have the same direction. The three couples in Fig. 30 are equal if $G_1 = G_2 = G_3$.

Resultant of two couples is a third couple, whose torque is the vector sum of the torques of the given couples.

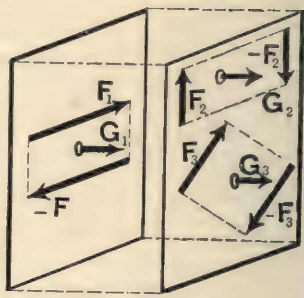


FIG. 30.

PROBLEMS.

1. Find the direction and magnitude of the resultant torque of three couples of equal magnitude the forces of which act along the edges of the bases of a right prism. The bases of the prism are equilateral triangles.

2. In the preceding problem let the forces have a magnitude of 15 pounds each, the length of the prism be 2 feet and the sides of the bases 10 inches.

3. In problem 1 suppose the prism to have hexagonal bases.

4. In problem 2 suppose the prism to be hexagonal.

5. A right circular cone, of weight W and angle 2α, is placed in a circular hole of radius r, cut in a horizontal table. Assuming the coefficient of friction between the cone and the table to be μ, find the least torque necessary to rotate the former about its axis.

47. Moment of a Force. — The most common method of giving a rigid body a motion of rotation is to put an axle through it and to apply to it a force which acts in a plane perpendicular to the axle. The rotation is produced by the couple formed by the applied force and the reaction of the axle. The torque due to the couple equals the product of the applied force by the shortest distance from the axle to the line of action of the force. It is often more con-

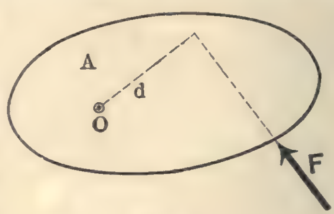


FIG. 31.

venient to disregard the reaction of the axle. When this is done the torque of the couple is called the *moment* of the force applied. Therefore the *moment of a force about an axis equals the product of the force by its lever-arm.* The lever-arm of a force is the shortest distance between the axis and the line of action of the force. In Fig. 31 the moment of **F** about the axis through the point O and perpendicular to the plane of the paper is

$$G = Fd, \tag{II}$$

where d is the lever-arm.

PROBLEMS.

1. Prove that the moment of a force about an axis equals the moment of its component which lies in a plane perpendicular to the axis.

2. Prove that the sum of the moments of the forces of a couple about any axis perpendicular to the plane of the couple is constant and equals the torque of the couple.

48. Degrees of Freedom of a Rigid Body. — A rigid body may have a motion of translation along each of the axes of a rectangular system of coördinates and at the same time it can have a motion of rotation about each of these axes. Therefore a rigid body has six degrees of freedom, three of translation and three of rotation. When one point in it is constrained to move in a plane the number of degrees of freedom is reduced to five. When the point is constrained to move in a straight line the number becomes four. When the point is fixed the body has only the three degrees of freedom of rotation. If two points are fixed the body can only rotate about the line joining the two points. Therefore its freedom is reduced to one degree. When a third point, which is not in the line determined by the other two, is fixed the body cannot move at all, that is, it has no freedom of motion.

49. The Law of Action and Reaction.—The law from which the conditions of equilibrium of a particle were obtained is a

universal law applicable to all bodies under all conditions; therefore it is applicable to rigid bodies as well as to single particles. But since rigid bodies may be subject to two distinct types of action the law may be stated in the following form.

The sum of all the linear and angular actions to which a body or a part of body is subject at any instant vanishes:

$$\Sigma(\mathbf{A}_l + \mathbf{A}_a) = 0. \tag{A'}$$

But since the two types of action are independent of each other the sum of each type must vanish when the combined sum vanishes. Therefore we can split the law into the following two sections.

To every linear action there is an equal and opposite linear reaction, or, the sum of all the linear actions to which a body or a part of body is subject at any instant vanishes:

$$\Sigma\mathbf{A}_l = 0. \tag{A$_l$}$$

To every angular action there is an equal and opposite angular reaction, or, the sum of all the angular actions to which a body or a part of body is subject at any instant vanishes:

$$\Sigma\mathbf{A}_a = 0. \tag{A$_a$}$$

50. Conditions of Equilibrium of a Rigid Body.— If we replace the term "linear action" in the first section of the law by the word "force" and the term "angular action" in the second section of the law by the word "torque" we obtain the two conditions which must be satisfied in order that a rigid body be in equilibrium. Thus, in order that a rigid body be in equilibrium the following conditions must be satisfied.

First. *The sum of all the forces acting upon the rigid body must vanish,* that is, if $\mathbf{F}_1, \mathbf{F}_2, \ldots \mathbf{F}_n$ denote all the forces acting upon the body then the vector equation

$$\mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n = 0 \tag{III}$$

must be satisfied.

Second. *The sum of all the torques acting upon the rigid body must vanish,* that is, if $\mathbf{G}_1$, $\mathbf{G}_2$, . . . $\mathbf{G}_n$ denote all the torques acting upon the body then the vector equation

$$\mathbf{G}_1 + \mathbf{G}_2 + \cdots + \mathbf{G}_n = 0 \tag{IV}$$

must be satisfied.

The following forms of the statement of these two conditions are better adapted for analysis.

First. *The algebraic sum of the components of all the forces along each of the axes of a rectangular system of coördinates must vanish,* that is,

$$\left.\begin{aligned} \Sigma X &\equiv X_1 + X_2 + \cdots + X_n = 0, \\ \Sigma Y &\equiv Y_1 + Y_2 + \cdots + Y_n = 0, \\ \Sigma Z &\equiv Z_1 + Z_2 + \cdots + Z_n = 0. \end{aligned}\right\} \tag{V$'$}$$

Second. *The algebraic sum of the components of all the torques about each of the axes of a system of rectangular coördinates must vanish,* that is,

$$\left.\begin{aligned} \Sigma G_x &\equiv G_x' + G_x'' + \cdots + G_x^{(n)} = 0, \\ \Sigma G_y &\equiv G_y' + G_y'' + \cdots + G_y^{(n)} = 0, \\ \Sigma G_z &\equiv G_z' + G_z'' + \cdots + G_z^{(n)} = 0. \end{aligned}\right\} \tag{VI$'$}$$

51. Coplanar Forces.— If two or more forces act in the same plane they are said to be *coplanar.* If a system of coplanar forces act in the xy-plane then the conditions of equilibrium reduce to the following equations:

$$\left.\begin{aligned} \Sigma X &\equiv X_1 + X_2 + \cdots + X_n = 0, \\ \Sigma Y &\equiv Y_1 + Y_2 + \cdots + Y_n = 0, \end{aligned}\right\} \tag{V}$$

$$\Sigma G_z \equiv F_1 d_1 + F_2 d_2 + \cdots + F_n d_n = 0, \tag{VI}$$

where d_1, d_2, . . . , d_n are the lever-arms of the forces $\mathbf{F}_1$, $\mathbf{F}_2$, . . . $\mathbf{F}_n$, respectively, about any axis which is perpendicular to the plane of the forces. The z-components of the forces and the x- and y-components of the moments vanish identically. Consequently they need not be considered.

52. Transmissibility of Force. — A force which acts upon a rigid body may be considered to be applied to any particle of the body which lies on the line of action of the force. In order to prove this statement consider the rigid body A, Fig. 32, which is in equilibrium under the action of the two

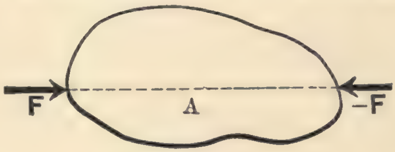


FIG. 32.

equal and opposite forces $\mathbf{F}$ and $-\mathbf{F}$. Now suppose we change the point of application of $\mathbf{F}$, without changing either its direction or its line of application. Evidently the equilibrium is not disturbed, because by moving $\mathbf{F}$ in its line of action we neither changed the sum of the forces nor the sum of their moments about any axis. Therefore the line of action of a force is of importance and not its point of application.

53. Internal Forces. — *Internal forces do not affect the equilibrium of a rigid body.* This is a direct consequence of the law of "action and reaction." Since by definition the internal forces are due to the interaction between the particles of the system these forces exist in equal and opposite pairs, therefore mutually annul each other.

ILLUSTRATIVE EXAMPLES.

1. A uniform beam rests with its lower end on smooth horizontal ground and its upper end against a smooth vertical wall. The beam is held from slipping by means of a string which connects the foot of the beam with the foot of the wall. Find the tensile force in the string and the reactions at the ends of the beam.

There are four forces acting upon the beam, i.e., the two reactions, $\mathbf{R}_1$ and $\mathbf{R}_2$, the tensile force $\mathbf{T}$ and the weight $\mathbf{W}$. Since both the ground and the wall are supposed to be smooth, $\mathbf{R}_1$ is normal to the ground, and $\mathbf{R}_2$

to the wall. Therefore denoting the lengths of the beam and the string by l and a, respectively, we have

$$\Sigma X = R_2 - T = 0,$$
$$\Sigma Y = R_1 - W = 0,$$
$$\Sigma G_{o'} = -R_2 l \sin\alpha + W\frac{l}{2}\cos\alpha = 0,$$

where $\Sigma G_{o'}$ denotes the sum of the moments of the forces about an axis through the point O' perpendicular to the xy-plane. Solving the last three equations we have

$$R_1 = W,$$
$$R_2 = \frac{W}{2}\cot\alpha$$
$$= \frac{W}{2}\frac{a}{\sqrt{l^2 - a^2}},$$

and
$$T = \frac{W}{2}\frac{a}{\sqrt{l^2 - a^2}}.$$

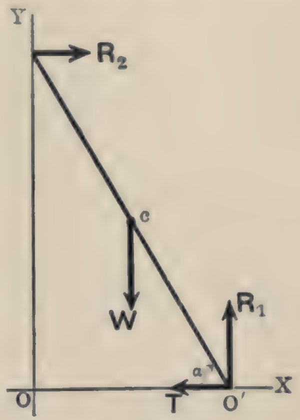


FIG. 33.

DISCUSSION. — It should be noticed that in taking the moments the axis was chosen through the point O' in order to eliminate the moments of as many forces as possible and thus to obtain a simple equation.

The reaction $\mathbf{R}_1$ is independent of the angular position of the beam and equals the weight $\mathbf{W}$. On the other hand $\mathbf{R}_2$ and $\mathbf{T}$ vary with α. When $\alpha = \frac{\pi}{2}$ both $\mathbf{R}_2$ and $\mathbf{T}$ vanish. As α is diminished from $\frac{\pi}{2}$ to 0, $\mathbf{R}_2$ and $\mathbf{T}$ increase indefinitely.

2. A ladder rests on rough horizontal ground and against a rough vertical wall. The coefficient of friction between the ladder and the ground is the same as that between the ladder and the wall. Find the smallest angle the ladder can make with the horizon without slipping.

There are three forces acting on the ladder, i.e., its own weight $\mathbf{W}$ and the two reactions $\mathbf{R}_1$ and $\mathbf{R}_2$. Replacing $\mathbf{R}_1$ and $\mathbf{R}_2$ by their components and writing the equations of equilibrium we obtain

$$\Sigma X = F_1 - N_2 = 0,$$
$$\Sigma Y = N_1 + F_2 - W = 0,$$
$$\Sigma G_{o'} = F_2 l\cos\alpha + N_2 l\sin\alpha - W\frac{l}{2}\cos\alpha = 0,$$

where α is the required angle.

We have further

$$\mu = \frac{F_1}{N_1} = \frac{F_2}{N_2}.$$

Solving these we get

$$F_1 = \frac{\mu}{1+\mu^2} W,$$

$$N_1 = \frac{1}{1+\mu^2} W,$$

$$R_1 = \frac{1}{\sqrt{1+\mu^2}} W,$$

$$F_2 = \frac{\mu^2}{1+\mu^2} W,$$

$$N_2 = \frac{\mu}{1+\mu^2} W,$$

$$R_2 = \frac{\mu}{\sqrt{1+\mu^2}} W,$$

$$\tan\alpha = \frac{1-\mu^2}{2\mu}.$$

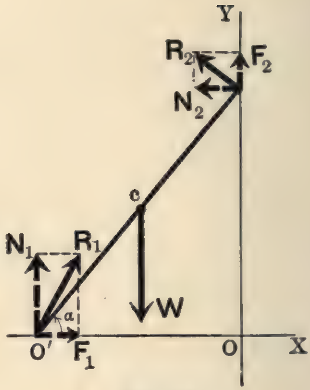


FIG. 34.

DISCUSSION. — The last expression gives the value of α for a given value of μ. When $\mu = 1$, $\alpha = 0$, therefore in this case the ladder will be in equilibrium at any angle between 0 and $\frac{\pi}{2}$ with the ground. Evidently this is true for any value of μ greater than unity.

3. Find the smallest force which, when applied at the center of a carriage wheel of radius a, will drag it over an obstacle.

The forces acting on the wheel are: its weight **W**, the required force **F**, and the reaction **R**. Since the first two meet at the center of the wheel, the direction of **R** must pass through the center also. Take the coördinate axes along and at right angles to **R**, as shown in Fig. 35, and let **F** make an angle θ with the x-axis. Then the equations of equilibrium become

$$\begin{aligned} \Sigma X &\equiv F\cos\theta - R + W\cos\alpha = 0, \\ \Sigma Y &\equiv F\sin\theta - W\sin\alpha = 0, \\ \Sigma G_{o'} &\equiv W\cdot a\sin\alpha - F\sin\theta\cdot a = 0. \end{aligned}$$

From either of the last two equations we get

$$F = \frac{\sin\alpha}{\sin\theta} W.$$

Since W and α are fixed F can be changed only by changing θ. Therefore the minimum value of F is given by the maximum value of $\sin\theta$, i.e., $\theta = \frac{\pi}{2}$, which makes

$$F = W\sin\alpha.$$

From the figure we obtain $\cos \alpha = \dfrac{a-h}{a}$,

therefore $$\sin \alpha = \frac{1}{a}\sqrt{h(2a-h)},$$

and $$F = \frac{W}{a}\sqrt{h(2a-h)}.$$

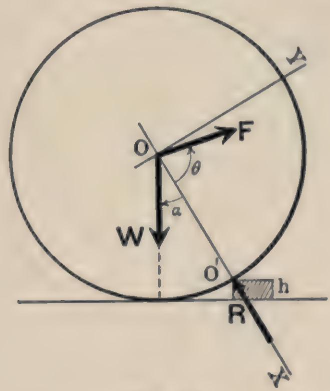


FIG. 35.

Since $\cos \theta = 0$ the first equation of equilibrium gives

$$R = W \cos \alpha$$
$$= \frac{a-h}{a} W.$$

DISCUSSION. — It will be observed that the first two of the equations of equilibrium are sufficient to solve the problem.

When h is zero, $F = 0$ and $R = W$. On the other hand when $h = a$, $F = W$ and $R = 0$.

PROBLEMS.

1. Prove that the true weight of a body is the geometric mean between the apparent weights obtained by weighing it in both pans of a false balance.

2. A uniform bar weighing 10 pounds is supported at the ends. A weight of 25 pounds is suspended from a point 20 cm. from one end. Find the pressure at the supports if the length of the bar is 50 cm.

3. A uniform rod which rests on a rough horizontal floor and against a smooth vertical wall is on the point of slipping. Find the reactions at the two ends of the rod.

4. A body is suspended from the middle of a uniform rod which passes over two fixed supports 6 feet apart. In moving the body 6 inches nearer to one of the supports the pressure on the support increases by 100 pounds. What is the weight of the body if 5 pounds is the weight of the rod?

5. A uniform rod of length a and weight W is suspended by two strings having lengths l_1 and l_2. The lower ends of the strings are attached to the ends of the rod, while the upper ends are tied to a peg. Find the tensile force in the strings.

6. A safety valve consists of a cylinder with a plunger attached to a uniform bar hinged at one end. The plunger has a diameter of $\frac{1}{2}$ inch and is attached to the bar at a distance of 1 inch from the hinge. The bar is 2 feet long and weighs 1 pound. How far from the hinge must a slide-weight of 2 pounds be set if the steam is to blow off at 120 pounds per square inch?

7. The two legs of a stepladder are hinged at the top and connected at the middle by a string of negligible mass. Find the tensile force in the string and the pressure on the hinges when the ladder stands on a smooth plane. The weight of the ladder is W, the length of its legs l, and the length of the string a.

8. A uniform rod rests on two smooth inclined planes making angles of α_1 and α_2 with the horizon. Find the angle which the rod makes with the horizon and the pressure on the planes.

9. A rectangular block is placed on a rough inclined plane whose inclination is gradually increased. If the block begins to slide and to turn about its lowest edge simultaneously find the coefficient of friction.

10. A uniform rod rests with one end against a rough vertical wall and the other end connected to a point in the wall by a string of equal length. Show that the smallest angle which the string can make with the wall is $\tan^{-1}\left(\frac{3}{\mu}\right)$.

11. A uniform rod is suspended by a string which is attached to the ends and is slung over a smooth peg. Show that in equilibrium the rod is either horizontal or vertical.

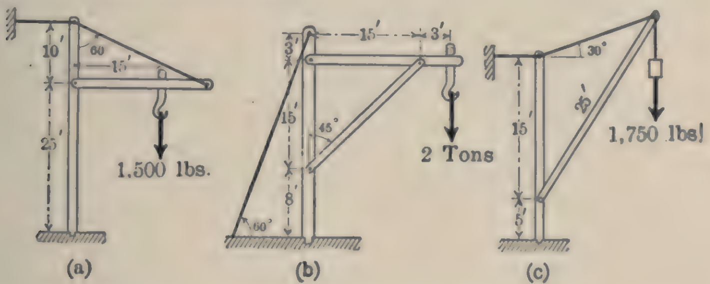
12. A ladder 25 feet long and weighing 50 pounds rests against a vertical wall making 30° with it. How high can a man weighing 150 pounds climb up the ladder before it begins to slip? The coefficient of friction is 0.5 at both ends of the ladder.

13. A rod of negligible weight rests wholly inside a smooth hemispherical bowl of radius r. A weight W is clamped on to the rod at a point whose distances from the ends are a and b. Show that the equilibrium

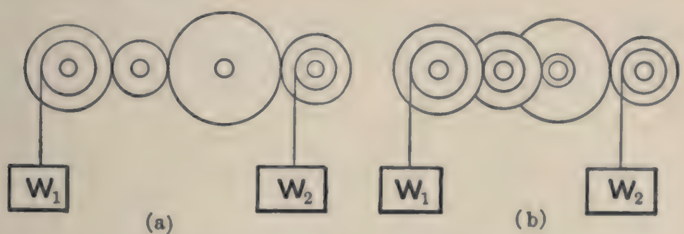
position of the rod is given by $\sin\theta = \dfrac{a-b}{2\sqrt{r^2-ab}}$, where θ is the angle it makes with the plane of the brim of the bowl which is horizontal.

14. Prove that when a rigid body is in equilibrium under the action of three forces their lines of action lie in the same plane and intersect at the same point.

15. Find the forces which tend to compress or extend the different members of the following cranes.



16. Supposing the weights of the following figures to be in equilibrium find their relative magnitudes. The circles which are tangent to other circles represent gears.



54. Resultant of a System of Forces Acting upon a Rigid Body.—We have already shown that the most general displacement of a rigid body consists of a translation along, and a rotation about, a certain line. Therefore such a displacement can be prevented by a single force opposed to the translation and a single torque opposed to the rotation. Thus a single force and a single torque can be found which will keep a rigid body in equilibrium against the action of any system of forces.

The resultant of a system of forces consists, therefore, of a single force and a single torque which, when reversed, will keep the rigid body in equilibrium against the action of the given system of forces.

55. Resultant of Coplanar Forces Acting upon a Rigid Body. — Let $\mathbf{F}_1$, $\mathbf{F}_2$, . . . $\mathbf{F}_n$ denote the given forces and let the *xy*-plane be their plane of action. Then, if **R**, **X**, and **Y** denote the resultant force and its components, respectively, we have

$$\left.\begin{aligned} X &= X_1 + X_2 + \cdots + X_n, \\ Y &= Y_1 + Y_2 + \cdots + Y_n, \end{aligned}\right\} \tag{VII}$$

$$R = \sqrt{X^2 + Y^2}, \tag{VIII}$$

and

$$\tan\theta = \frac{Y}{X}, \tag{IX}$$

where the terms in the right-hand members of the first two equations are the components of the given forces, and θ is the angle **R** makes with the x-axis.

On the other hand if $\mathbf{G}_o$ denotes the resultant torque and $d_1, d_2, \ldots, d_n$ denote the distances of the origin from the lines of action of the forces, then

$$G_o = F_1 d_1 + F_2 d_2 + \cdots + F_n d_n. \tag{X}$$

If we represent this torque by the moment of the resultant force about the z-axis, then

$$\left.\begin{aligned} RD &= F_1 d_1 + F_2 d_2 + \cdots + F_n d_n, \\ \text{or} \qquad D &= \frac{\Sigma(Fd)}{R}, \end{aligned}\right\} \tag{XI}$$

gives the distance of the line of action of the resultant force from the origin.

ILLUSTRATIVE EXAMPLE.

Find the resultant of the six forces acting along the sides of the hexagon of Fig. 36.

Taking the sum of the components along the x and y directions, we have

$$X = 2F + 3F\cos\frac{\pi}{3} - 2F\cos\frac{\pi}{3} - F - 2F\cos\frac{\pi}{3} + F\cos\frac{\pi}{3}$$
$$= F.$$
$$Y = 0 - 3F\sin\frac{\pi}{3} - 2F\sin\frac{\pi}{3} + 0 + 2F\sin\frac{\pi}{3} + F\sin\frac{\pi}{3}$$
$$= -F\sqrt{3}.$$
$$\therefore \quad R = \sqrt{F^2 + 3F^2}$$
$$= 2F$$
and $\qquad \tan\theta = -\sqrt{3}.$

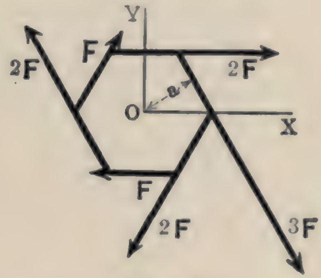


FIG. 36.

Therefore the resultant force has a magnitude $2F$ and makes an angle of $-60°$ with the x-axis.

Taking the moments about an axis through the center of the hexagon, we obtain

$$RD = (2F + 3F + 2F + F + 2F + F)a$$
$$= 11\,Fa,$$
therefore $\qquad D = 5.5\,a,$

where a is the distance of the center from the lines of action of the forces.

56. Resultant of a System of Parallel Forces. — Let **R** be the resultant of the parallel forces $\mathbf{F}_1$, $\mathbf{F}_2$, . . . , $\mathbf{F}_n$, which act upon a rigid body. Then, since the forces are parallel, the resultant force equals the algebraic sum of the given forces. Thus

$$R = F_1 + F_2 + \cdots + F_n,$$
and
$$RD = F_1d_1 + F_2d_2 + \cdots + F_nd_n.$$

Now take the z-axis parallel to the forces and let x_i and y_i denote the distances of $\mathbf{F}_i$ from the yz-plane and the xz-plane, respectively. Then the last equation may be split into two parts, one of which gives the moments about the x-axis and the other about the y-axis. Thus,

$$\left.\begin{aligned} R\bar{x} &= F_1x_1 + F_2x_2 + \cdots + F_nx_n, \\ R\bar{y} &= F_1y_1 + F_2y_2 + \cdots + F_ny_n, \end{aligned}\right\} \qquad \text{(XII)}$$

where $\bar{x}$ and $\bar{y}$ are the coördinates of the point in the xy-plane through which the resultant force passes. In other

words, $(\bar{x},\bar{y})$ is the point of application of the resultant force. The resultant force is evidently parallel to the given forces. The last two equations may be written in the following forms

$$\left.\begin{aligned}\bar{x} &= \frac{\Sigma Fx}{R},\\ \bar{y} &= \frac{\Sigma Fy}{R}.\end{aligned}\right\} \qquad \text{(XIII)}$$

ILLUSTRATIVE EXAMPLE.

Find the resultant of two parallel forces which act upon a rigid body in the same direction.

Let the y-axis be parallel to the forces.

Then $$R = F_1 + F_2,$$

and $$\bar{x} = \frac{F_1x_1 + F_2x_2}{F_1 + F_2},$$

or $$\frac{F_1}{F_2} = \frac{x_2 - \bar{x}}{\bar{x} - x_1}.$$

But since $x_2 - \bar{x}$ and $\bar{x} - x_1$ are the distances of $\mathbf{F}_2$ and $\mathbf{F}_1$ from $\mathbf{R}$, we have

$$\frac{F_1}{F_2} = \frac{d_2}{d_1},$$

or $$F_1d_1 = F_2d_2.$$

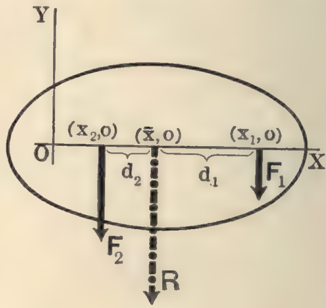


FIG. 37.

Therefore the distances of the resultant from the given forces are inversely proportional to the magnitudes of the latter.

PROBLEMS.

1. Find the resultant force and the resultant torque due to the forces P, $2\,P$, $4\,P$ and $2\,P$ which act along the sides of a square, taken in order.

2. Three forces are represented in magnitude and line of action by the sides of an equilateral triangle. Find the resultant force, taking the directions of one of the forces opposite to that of the other two.

3. The lines of action of three forces form a right isosceles triangle of sides a, a, and $a\sqrt{2}$. The magnitudes of the forces are proportional to the sides of the triangle. Find the resultant force.

4. The sum of the moments of a system of coplanar forces about any three points, which are not in the same straight line, are the same. Show that the system is equivalent to a couple.

5. Three forces are represented in magnitude, direction, and line of action by the sides of a triangle taken in order; prove that their resultant is a couple the torque of which equals, numerically, twice the area of the triangle.

6. Three forces act along the sides of an equilateral triangle; find the condition which will make their resultant pass through the center of the triangle.

FRICTION ON JOURNALS AND PIVOTS.

57. Friction on Journal Bearings. — If the horizontal shaft of Fig. 38 fits perfectly in its bearings the friction which comes into play is a sliding friction, therefore the laws of sliding friction may be assumed to hold good. The most important of these laws is: the frictional force which comes into play is proportional to the normal reaction, that is, in the relation

$$F = \mu N,$$

μ is independent of N. We will assume therefore that this law holds at each point of the surface of contact and thus reduce the problem under discussion to one of sliding friction. There is an important difference, however, between the problem under discussion and the problems on friction which we have already discussed. In the present problem the normal reaction is not the same at all the points of the surfaces in contact. We must apply, therefore, the laws of friction to small elements of surfaces of contact over which the normal reaction may be considered to be constant.

Let the element of surface be a strip, along the length of the shaft, which subtends an angle $d\theta$ at the axis of the shaft. Further let $d\mathsf{N}$ be the normal reaction over this element of surface, and $d\mathsf{F}$ be the corresponding frictional force; then we have

$$\begin{aligned} dF &= \mu\, dN \\ &= \mu p \cdot l \cdot a\, d\theta, \end{aligned}$$

where p is the normal reaction per unit area or the pressure, a is the radius of the shaft, and l the length of the bearing.

Therefore the total frictional force and the total frictional torque are, respectively,

$$F = \mu al \int_0^\pi p \, d\theta$$

and
$$G = \mu a^2 l \int_0^\pi p \, d\theta.$$

In order to carry out the integral of the foregoing expressions we have to make some assumption with regard to the nature

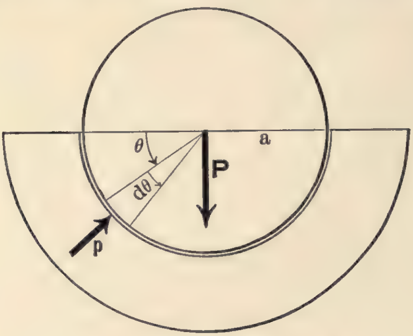


Fig. 38.

of dependence of p upon θ. But whatever the relation between p and θ it is obvious that the sum, over all the surfaces of contact, of the vertical component of the normal reaction must equal the load which rests upon the bearings. If P denotes this load, then p must satisfy the condition

$$P = \int_0^A p \sin \theta \cdot dA$$

$$= al \int_0^\pi p \sin \theta \, d\theta,$$

where A is the total area of contact.

ILLUSTRATIVE EXAMPLE.

The normal pressure on the bearings is given by the relation $p = p_0 \sin \theta$; find the total frictional force and the total frictional torque.

Substituting the given value of p in the expression for F we obtain

$$\begin{aligned} F &= \mu a l p_0 \int_0^\pi \sin\theta \, d\theta \\ &= 2\,\mu a l p_0. \end{aligned}$$

In order to determine p_0 in terms of the total load on the bearings we make p satisfy the condition

$$P = al \int_0^\pi p \sin\theta \, d\theta.$$

Substituting the given value of p in the right-hand member of the preceding equation we have

$$\begin{aligned} P &= alp_0 \int_0^\pi \sin^2\theta \, d\theta \\ &= \frac{\pi a l p_0}{2}, \end{aligned}$$

or

$$p_0 = \frac{2P}{\pi a l}.$$

Therefore

$$F = \frac{4\mu}{\pi} P$$

and

$$G = \frac{4\mu}{\pi} aP.$$

It will be observed that the total frictional force varies with the load and is independent of the radius and of the length of the bearing; in other words it is independent of the area of contact.

PROBLEMS.

1. Supposing the normal pressure to be the same at every point of the surfaces of contact, derive the expressions for the total frictional force and the resisting torque due to friction.

2. Supposing the vertical component of the total reaction at every point of the surfaces of contact to be constant, derive the expressions for the total frictional force and the resisting torque due to friction.

3. Derive expressions for the total frictional force and the resisting torque upon the assumption that the normal pressure is given by the relation $p = p_0 \sin^2\theta$.

58. Friction on Pivots. — The problem of friction on pivots also is a problem of sliding friction. The feature

which distinguishes the pivot from the journal bearing is this: in the former the lever arm of the frictional force varies from point to point, while in the latter it is constant and equals the radius of the shaft.

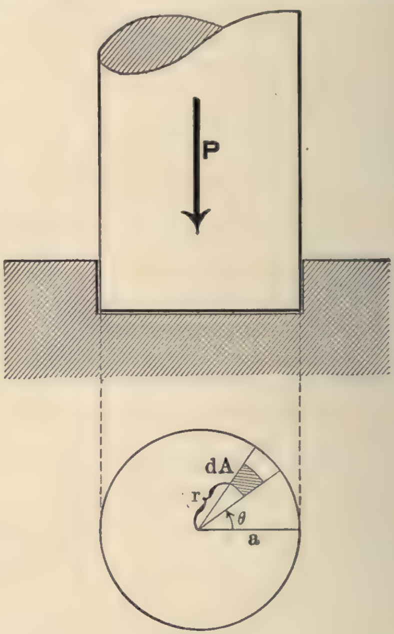


FIG. 39.

Let dN be the normal reaction upon dA, an element of area at the base of the flat-end pivot of Fig. 39; then if dF denotes the corresponding frictional force, we have

$$\begin{aligned} dF &= \mu\, dN \\ &= \mu p\, dA, \end{aligned}$$

where p is the normal pressure. Evidently p is constant; therefore we can write

$$\begin{aligned} F &= \mu p \int_0^{\pi a^2} dA \\ &= \pi a^2 \mu p. \end{aligned}$$

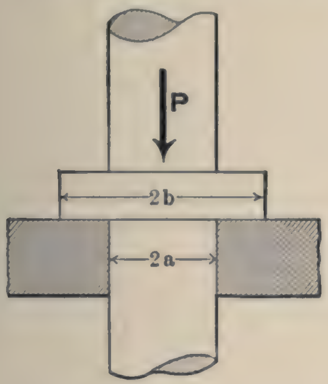
The expression for the resisting torque due to the friction is obtained as follows:

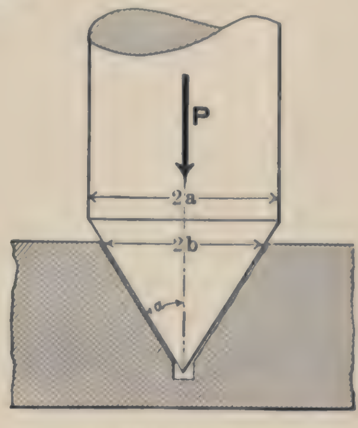
$$\begin{aligned} G &= \int_0^F r \cdot dF \\ &= \int_0^A r \cdot \mu p\, dA \\ &= \int_0^a \int_0^{2\pi} r\mu p \cdot r\, d\theta \cdot dr \\ &= \pi \mu p \int_0^a r^2\, dr \\ &= \tfrac{2}{3}\pi a^3 \mu p \\ &= \tfrac{2}{3} a\mu P, \end{aligned}$$

where P is the total load on the pivot.

PROBLEMS.

1. Derive an expression for the resisting torque due to friction in the collar-bearing pivot of the adjoining figure.

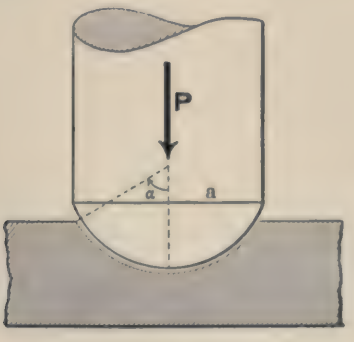




2. Supposing the normal pressure to be constant, derive an expression for the resisting torque due to friction in the conical pivot of the adjoining figure.

3. In the preceding problem suppose the vertical component of the normal pressure to be constant.

4. In problem 2 suppose the horizontal component of the normal pressure to be constant.



5. Taking the normal pressure to be constant derive an expression for the resisting torque, due to friction in the spherical pivot of the adjoining figure.

6. Prove that the resisting torque due to friction is greater for a hollow pivot than for a solid pivot, provided that the load and the load per unit area are the same in both cases.

7. Show that the resisting torque due to friction for a hemispherical pivot is about 2.35 times as large as that for a flat end pivot.

ROLLING FRICTION.

59. Coefficient of Rolling Friction. — Consider a cylinder, Fig. 40, which is in equilibrium on a rough horizontal plane under the action of a force **S**. In addition to this force the cylinder is acted upon by its weight and by the reaction of the plane. Applying the conditions of equilibrium we obtain

$$\Sigma X \equiv S - F = 0,$$
$$\Sigma Y \equiv -W + N = 0,$$
$$\Sigma G_0 \equiv ND - Sd = 0,$$

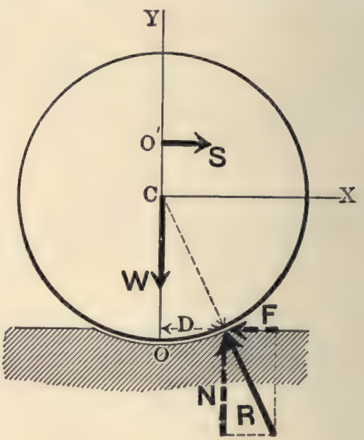


FIG. 40.

where **F** and **N** are the components of **R**, the reaction of the plane, while D and d are, respectively, the distances of the points of application of **R** and **S** from the point O, about which the moments are taken. These equations give us

$$R = \sqrt{F^2 + N^2}$$
$$= \sqrt{S^2 + W^2}, \tag{1}$$

and

$$D = \frac{S}{W}d. \tag{2}$$

If the cylinder is just on the point of motion

$$F = \mu N,$$

and consequently

$$\mu = \frac{S}{W}. \tag{3}$$

Combining (2) and (3), we obtain

$$D = \mu d. \tag{XIV}$$

The distance D is called the *coefficient of rolling friction.* Equation (XIV) states, therefore, that the coefficient of the rolling friction equals the coefficient of the sliding friction times the distance of the point of contact from the line of action of the force which urges the body to roll.

60. Friction Couple. — It is evident from the above equations that a change in the value of d does not affect the values of N and F, consequently it does not change the value of μ. This is as it should be, since, according to the laws of sliding friction, μ depends only upon the nature of the surfaces in contact. A change in d, however, changes the value of D; in other words, it changes the point of application of **R**. When $d = 0$, that is, when **S** is applied at the point of contact, $D = 0$, in which case the body is urged to slide only. But when d is not zero the force **S** not only urges the body to slide but also to roll; therefore, in addition to the resisting force **F**, a resisting torque comes into play. This torque, which is due to the couple formed by N and W, is called the *friction couple.*

PROBLEMS.

1. A gig is so constructed that when the shafts are horizontal the center of gravity of the gig is over the axle of the wheels. The gig rests on perfectly rough horizontal ground. Find the least force which, acting at the ends of the shafts, will just move the gig.

2. Find the smallest force which, acting tangentially at the rim of a flywheel, will rotate it. The weight and the radius of the flywheel, the radius of the shaft, and the coefficient of friction between the shaft and its bearings are supposed to be known.

3. A flywheel of 500 pounds weight is brought to the point of rotation by a weight of 10 pounds suspended by means of a string wound around its rim. Find the coefficient of friction between the axle and its bearings. The diameters of the wheel and the axle are 10 feet and 8 inches, respectively.

4. A wheel of radius a and weight W stands on rough horizontal ground. If μ is the coefficient of friction between the wheel and the ground find the smallest weight which must be suspended at one end of the horizontal diameter in order to move the wheel.

GENERAL PROBLEMS.

1. A table of negligible weight has three legs, the feet forming an equilateral triangle. Find the proportion of the weight carried by the legs when a particle is placed on the table.

2. A rectangular board is supported in a vertical position by two smooth pegs in a vertical wall. Show that if one of the diagonals is parallel to the line joining the pegs the other diagonal is vertical.

3. A uniform rod rests with its two ends on smooth inclined planes making angles α and β with the horizon. Where must a weight equal to that of the rod be clamped in order that the rod may rest horizontally?

4. A uniform ladder rests against a rough vertical wall. Show that the least angle it can make with the horizontal floor on which it rests is given by $\tan \theta = \dfrac{1 - \mu\mu'}{2\,\mu}$, where μ and μ' are the coefficients of friction for the floor and the wall, respectively.

5. A uniform rod is suspended by two equal strings attached to the ends. In position of equilibrium the strings are parallel and the bar is horizontal. Find the torque which will turn the bar, about a vertical axis, through an angle θ and keep it in equilibrium at that position.

6. The line of hinges of a door makes an angle α with the vertical. Find the resultant torque when the door makes an angle β with its equilibrium position.

7. The lines of action of four forces form a quadrilateral. If the magnitude of the forces are a, b, c, d times the sides of the quadrilateral find the conditions of equilibrium.

8. A force acts at the middle point of each side of a plane polygon. Each force is proportional to the length of the side it acts upon and is perpendicular to it. Prove that the polygon will be in equilibrium if all the forces are directed towards the inside of the polygon.

9. A force acts at each vertex of a plane convex polygon in a direction parallel to one of the sides forming the vertex. Show that if the forces are proportional to the sides to which they are parallel and if their directions are in a cyclic order their resultant is a couple.

10. A uniform chain of length l hangs over a rough horizontal cylinder of radius a. Find the length of the portions which hang vertically when

the chain is on the point of motion under its own weight, (1) when a is negligible compared with l, (2) when it is not negligible compared with l.

11. Two equal weights are attached to the extremities of a string which hangs over a rough horizontal cylinder. Find the least amount by which either weight must be increased in order to start the system to move. The weight of the string is negligible.

12. Three cylindrical pegs of equal radius and roughness are placed at the vertices of a vertical equilateral triangle the two lower corners of which are in the same horizontal line. A string of negligible weight is attached to two weights and slung over the pegs. Find the ratio of the weights if they are on the point of motion.

13. A sphere laid upon a rough inclined plane of inclination α is on the point of sliding. Show that the coefficient of friction is $\frac{2}{7}\tan\alpha$.

14. A uniform ring of weight W hangs on a rough peg. A bead of weight w is fixed on the ring. Show that if the coefficient of friction between the ring and the peg is greater than $\dfrac{W}{\sqrt{W^2 + 2wW}}$ the ring will be in equilibrium whatever the position of the bead with respect to the peg.

15. A uniform rod is in equilibrium with its extremities on the interior of a rough vertical hoop. Find the limiting position of the rod.

16. A weight W is suspended from the middle of a cord whose ends are attached to two rings on a horizontal pole. If w be the weight of each ring, μ the coefficient of friction, and l the length of the cord, find the greatest distance apart between the rings compatible with equilibrium.

CHAPTER IV.

EQUILIBRIUM OF FLEXIBLE CORDS.

61. Simplification of Problems. — The simplest phenomenon in nature is the result of innumerable actions and reactions. The consideration of all the factors which contribute to any natural phenomenon would require unlimited analytical power. Fortunately the factors which enter into dynamical problems are not all of equal importance. Often the influence of one or two predominate, so that the rest can be neglected without an appreciable departure from the actual problem. Any one who attempts to solve a physical problem must recognize this fact and use it to advantage by representing the actual problem by an ideal one which has only the important characteristics of the former. This was done in the last two chapters in which bodies were treated as single particles and rigid bodies, and the problems were thereby simplified without changing their character.

The same procedure will be followed in discussing the equilibrium of flexible cords, such as belts, chains, and ropes. These bodies will be represented by an ideal cord of negligible cross-section and of perfect flexibility. The solution of the idealized problems gives us a close enough approximation for practical purposes. If, however, closer approximation is desired smaller factors, such as the effects of thickness and imperfect flexibility, may be taken into account.

62. Flexibility. — A cord is said to be *perfectly flexible* if it offers no resistance to bending; in other words, in a perfectly flexible cord there are no internal forces which act in a direction perpendicular to its length.

63. Suspension Bridge Problem. — The following are the important features of a suspension bridge which should be considered in order to simplify the problem:

1. The weights of the cables and of the chains are small compared with that of the road-bed.

2. The road-bed is practically horizontal.

3. The distribution of weight in the road-bed may be considered to be uniform.

We can, therefore, obtain a sufficiently close approximation if we consider an ideal bridge in which the cable and the chains have no weight and the distribution of weight in the road-bed is uniform in the horizontal direction. With these simplifications consider the forces acting upon that part of the cable which is between the lowest point and any point P, Fig. 41.

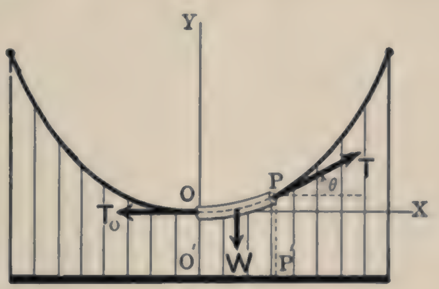


FIG. 41.

The forces are: The tensile force T_0, which acts horizontally at O. The tensile force T, which acts along the tangent to the curve at P. The weight of that part of the bridge which is between O and P. If w be the weight per unit length of the road-bed and x denotes the length $O'P'$, then the third force becomes $\mathsf{w}x$.

Therefore the conditions of equilibrium give

$$\Sigma X \equiv -T_0 + T\cos\theta = 0; \quad \therefore \quad T\cos\theta = T_0. \tag{1}$$

$$\Sigma Y \equiv -wx + T\sin\theta = 0; \quad \therefore \quad T\sin\theta = wx. \tag{2}$$

It is evident from equation (1) that the horizontal component of the tensile force is constant and equals T_0. Squaring equations (1) and (2) and adding we get

$$T^2 = T_0^{\,2} + w^2x^2. \tag{3}$$

Thus we see that the smallest value of T corresponds to $x = 0$ and equals T_0, while its greatest value corresponds to the greatest value of x. If D denotes the span of the bridge then the greatest value of T, or the tensile force of the cable at the piers, is

$$T_m = \sqrt{T_0^{\,2} + \frac{w^2D^2}{4}}.$$

In order to find the equation of the curve which the cable assumes we eliminate T between equations (1) and (2). This gives

$$\tan\theta = \frac{w}{T_0}x. \tag{4}$$

Substituting $\frac{dy}{dx}$ for $\tan\theta$ and integrating we get

$$y = \frac{1}{2}\frac{w}{T_0}x^2 + c,$$

where c is the constant of integration.

But with the axes we have chosen, $y = 0$ when $x = 0$, therefore $c = 0$. Thus the equation of the curve is

$$y = \frac{w}{2\,T_0}x^2, \tag{5}$$

which is the equation of a parabola.

Dip of the Cable. — Let H be the height of the piers above the lowest point of the cable. Then for $x = \frac{D}{2}$, $y = H$, therefore

$$H = \frac{w}{8\,T_0}D^2. \tag{6}$$

It is evident from the last equation that the greater the tension the less is the sag.

PROBLEM. A bridge is supported by two suspension cables. The bridge has a weight of 1.5 tons per horizontal foot and has a span of 400 feet. Supposing the dip of the bridge to be 50 feet find the values of the tensile force at the lowest and highest points of the cable.

64. Equilibrium of a Uniform Flexible Cord which is Suspended from Its Ends. — The problem is to determine the nature of the curve which a perfectly uniform and flexible cable will assume when suspended from two points. Let AOB, Fig. 42, be the curve. Consider the equilibrium of that part of the cable which is between the lowest point O and any other point P. The part of the cable which is under consideration is acted upon by the following three forces:

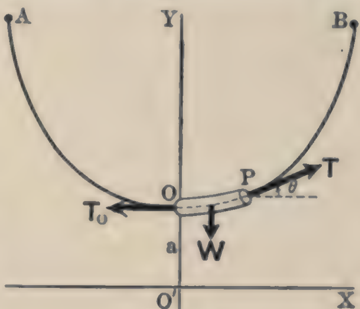


FIG. 42.

The tensile force at the point O, $\mathbf{T}_0$.

The tensile force at the point P, $\mathbf{T}$.

The weight of the cable between the points O and P.

Since the cable is perfectly flexible $\mathbf{T}_0$ and $\mathbf{T}$ are tangent to the curve. Therefore we have

$$\Sigma X \equiv -T_0 + T\cos\theta = 0, \quad \text{or} \quad T\cos\theta = T_0, \tag{1}$$

$$\Sigma Y \equiv -ws + T\sin\theta = 0, \quad \text{or} \quad T\sin\theta = ws, \tag{2}$$

where w is the weight per unit length of the cable and s is the length of OP.

Squaring equations (1) and (2) and adding we obtain

$$T^2 = T_0^2 + w^2s^2. \tag{3}$$

Eliminating T between equations (1) and (2) we get

$$s = \frac{T_0}{w}\tan\theta, \tag{4}$$

which is the intrinsic equation of the curve.

In order to express equation (4) in terms of rectangular coördinates we replace $\tan\theta$ by $\dfrac{dy}{dx}$ and obtain

$$s = \frac{T_0}{w} \cdot \frac{dy}{dx}. \tag{5}$$

But $ds^2 = dx^2 + dy^2$, therefore eliminating dx between this equation and equation (5) and separating the variables

$$dy = \frac{s\,ds}{\sqrt{s^2 + a^2}}, \tag{6}$$

and then integrating

$$y = \sqrt{s^2 + a^2} + c,$$

where $a = \dfrac{T_0}{w}$ and c is the constant of integration.

Let the x-axis be so chosen that when $s = 0$, $y = a$, then $c = 0$. Therefore

$$y = \sqrt{s^2 + a^2}, \quad \text{or} \quad s = \sqrt{y^2 - a^2}. \tag{7}$$

Differentiating equation (7), squaring and replacing ds^2 by $(dx^2 + dy^2)$ we have

$$dx^2 + dy^2 = \frac{y^2\,dy^2}{y^2 - a^2}.$$

Solving for dx,

$$\left.\begin{aligned} dx &= -\frac{a}{\sqrt{y^2 - a^2}}\,dy \\ &= -\frac{a\,dy}{\sqrt{-1}\,\sqrt{a^2 - y^2}} \\ &= -\frac{a\,dy}{i\sqrt{a^2 - y^2}}, \end{aligned}\right\} \tag{8}$$

where $i = \sqrt{-1}$. Integrating equation (8) we get

$$\frac{ix}{a} = \cos^{-1}\frac{y}{a} + c'.$$

But $y = a$, when $x = 0$, therefore $c' = 0$. Thus we get

$$y = a \cos \frac{ix}{a}, \tag{9}$$

$$= a \cosh \frac{x}{a},^* \tag{10}$$

$$= \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right), \tag{11}$$

$$= \frac{T_0}{2\,w}\left(e^{\frac{wx}{T_0}} + e^{-\frac{wx}{T_0}}\right), \tag{12}$$

which are different forms of the equation of a *catenary*.

DISCUSSION.—Expanding equation (12) by Maclaurin's Theorem† we obtain

$$y = a\left[1 + \frac{1}{2}\left(\frac{x}{a}\right)^2 + \frac{1}{24}\left(\frac{x}{a}\right)^4 + \cdot \cdot \cdot\right]. \tag{13}$$

In the neighborhood of the lowest point of the cable the value of x is small, therefore in equation (13) we can neglect all the terms which contain powers of x higher than the second. Thus the equation

$$y = a + \frac{x^2}{2\,a} \tag{14}$$

represents, approximately, the curve in the neighborhood of the lowest point. It will be observed that (14) is the equation of a parabola. This result would be expected since the curve is practically straight in the neighborhood of O and consequently the horizontal distribution of mass is very nearly constant, which is the important feature of the Suspension Bridge problem.

The nature of those parts of the curve which are removed from the lowest point may be studied by supposing x to be large. Then since $e^{-\frac{x}{a}}$ becomes negligible equation (11) reduces to

$$y = \frac{a}{2}e^{\frac{x}{a}}, \tag{15}$$

* See Appendix AVIII. † See Appendix AV.

The curve, Fig. 43, defined by equation (15) is called an *exponential curve.* It has an interesting property, namely, its ordinate is doubled every time a constant value P is added to its abscissa. This constant is called the *half-value period* of the curve. The value of P may be determined in the following manner. By the definition of P and from equation (15) we have

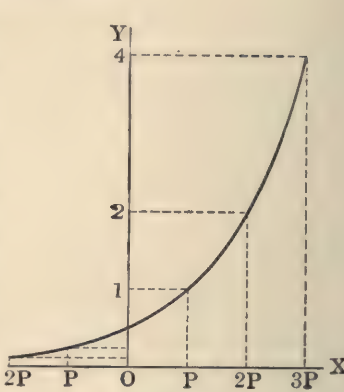


Fig. 43.

$$2\,y = \frac{a}{2} e^{\frac{x+P}{a}}. \tag{16}$$

Dividing equation (16) by equation (15) we get

$$2 = e^{\frac{P}{a}},$$

or

$$P = a \log_e 2.$$

Length of Cable.—In order to find the length in terms of the span eliminate y between equations (7) and (11). This gives

$$s = \frac{a}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right) \tag{17}$$

$$= x + \frac{1}{2 \cdot 3}\frac{x^3}{a^2} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5}\frac{x^5}{a^3} + \cdots, \tag{18}$$

where the right member of equation (18) is obtained by expanding the right-hand member of equation (17) by Maclaurin's Theorem.

If D and L denote the span and the length of the cable, respectively, we have $s = \frac{1}{2} L$ when $x = \frac{1}{2} D$. Therefore substituting these values of s and x in equation (18) and replacing a by its value we obtain

$$L = 2\left(\tfrac{1}{2} D + \frac{1}{48}\frac{w^2}{T_0{}^2} D^3 + \cdots\right). \tag{19}$$

When the cable is stretched tight T_0 is large compared with w. Therefore the higher terms of the series may be neglected and equation (19) be put in the following approximate form.

$$L = D\left(1 + \frac{1}{24}\frac{w^2}{T_0^2}D^2\right). \tag{20}$$

Hence the increase in length due to sagging is $\frac{1}{24}\frac{w^2}{T_0^2}D^3$, approximately.

PROBLEMS.

1. A perfectly flexible cord hangs over two smooth pegs, with its ends hanging freely, while its central part hangs in the form of a catenary. If the two pegs are on the same level and at a distance D apart, show that the total length of the string must not be less than De, in order that equilibrium shall be possible, where e is the natural logarithmic base.

2. In the preceding problem show that the ends of the cord will be on the x-axis.

3. Supposing that a telegraph wire cannot sustain more than the weight of one mile of its own length, find the least and the greatest sag allowable in a line where there are 20 poles to the mile.

4. Find the actual length of the wire per mile of the line in the preceding problem.

5. The width of a river is measured by stretching a tape over it. The middle point of the tape touches the surface of the water while the ends are at a height H from the surface. If the tape reads S, show that the width of the river is approximately $\sqrt{\frac{S^2 - H^2}{2}}$.

6. Show that the cost of wire and posts of a telegraph line is minimum if the cost of the posts is twice that of the additional length of wire required by sagging. The posts are supposed to be evenly spaced and large in number.

7. A uniform cable which weighs 100 tons is suspended between two points, 500 feet apart, in the same horizontal line. The lowest point of the cable is 40 feet below the points of support. Find the smallest and the greatest values of the tensile force.

8. In the preceding problem find the length of the cable.

65. Friction Belts. — The flexible cord AB, Fig. 44, is in equilibrium under the action of three forces, namely, $\mathbf{T}_0$

and **T**, which are applied at the ends of the cord, and the reaction of the rough surface of C, with which it is in contact. It is desired to find the relation between T_0 and T when the cord is just on the point of motion towards $\mathbf{T}_0$.

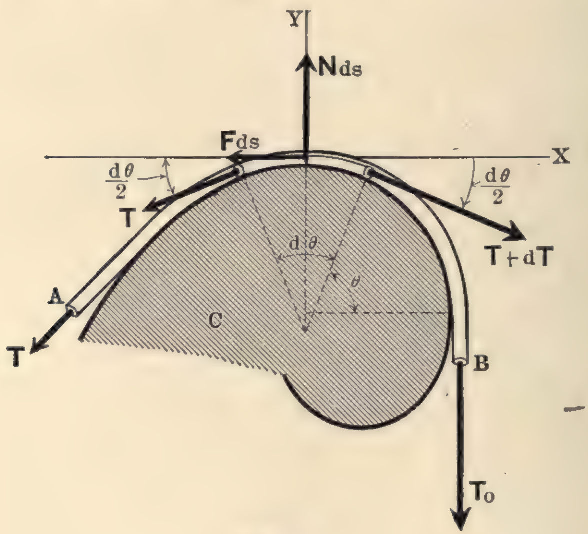


Fig. 44.

Consider the equilibrium of an element of that part of the cord which is in contact with the surface. The element is acted upon by the following three forces:

The tensile force in the cord to the right of the element.

The tensile force in the cord to the left of the element.

The reaction of the surface.

Let the tensile force to the left of the element be denoted by **T**, then the tensile force to the right may be denoted by $\mathbf{T} + d\mathbf{T}$. On the other hand if **R** denotes the reaction of the surface per unit length of the cord, the reaction on the element is **R** ds, where ds is the length of the element. We will, as usual, replace **R** by its frictional component **F** and its normal component **N**.

Taking the axes along the tangent and the normal through the middle point of the element and applying the conditions of equilibrium we obtain

$$\Sigma X \equiv (T + dT) \cos \frac{d\theta}{2} - T \cos \frac{d\theta}{2} - F\,(-ds)^* = 0,$$

$$\Sigma Y \equiv N\,ds - T \sin \frac{d\theta}{2} - (T + dT) \sin \frac{d\theta}{2} = 0,$$

or
$$dT \cos \frac{d\theta}{2} + F\,ds = 0,$$

and
$$N\,ds - 2\,T \sin \frac{d\theta}{2} - dT \sin \frac{d\theta}{2} = 0,$$

where $d\theta$ is the angle between the two tensile forces which act at the ends of the element. But since the cord is supposed to be perfectly flexible the tensile forces are tangent to the surface of contact. Therefore θ is the angle between the tangents, and consequently the angle between the normals, at the ends of the element. As an angle becomes indefinitely small its cosine approaches unity and its sine approaches the angle itself,† therefore we can make the substitutions

$$\cos \frac{d\theta}{2} = 1 \text{ and } \sin \frac{d\theta}{2} = \frac{d\theta}{2}$$

in the last two equations, and obtain

$$dT + F\,ds = 0, \tag{1}$$

and
$$N\,ds - T\,d\theta + \tfrac{1}{2}\,dT\,d\theta = 0. \tag{2}$$

Neglecting the differential of the second order in equation (2) and then eliminating ds between equations (1) and (2) we get

$$\frac{dT}{T} = -\frac{F}{N}\,d\theta = -\mu\,d\theta, \tag{3}$$

where μ is the coefficient of friction. Integrating the last

* The negative sign in $F\,(-ds)$ indicates the fact that **F** and ds are measured in opposite directions.

† See Appendix Avi.

equation and passing from the logarithmic to the exponential form, we have

$$T = ce^{-\mu\theta},$$

where c is the constant of integration. If θ is measured from the normal to the surface at the point where the right-hand side of the cord leaves contact we obtain the initial condition, $T = T_0$ when $\theta = 0$, which determines c. Applying this condition to the last equation we have

$$T = T_0e^{-\mu\theta}. \tag{4}$$

DISCUSSION. — Equation (4) gives the relation between the values of the tensile force at any two points of the cord. It must be observed that θ is measured in the same direction as **F**; in other words, opposite the direction towards which the cord is urged to move. Therefore T or T_0 has the larger value according to whether θ is positive or negative. As a concrete example suppose a weight W to be suspended from the right-hand end of the cord and to be held in equilibrium by a force F applied at the left-hand end. If F is just large enough to prevent W from falling then the cord will be on the point of moving to the right, therefore θ is measured in the counter-clockwise direction and is positive. In this case

$$F = We^{-\mu\theta}.$$

In case F is just large enough to start W to move up, then θ is measured in the clockwise direction and is negative. Therefore

$$F = We^{\mu\theta}.$$

The value of T drops very rapidly with the increase of θ. This fact is made clear by drawing the graph of equation (4), Fig. 45. The graph may be constructed easily by making use of the half-value period of the curve. If P denotes the period, then, by definition, the ordinate is reduced to one-half its value every time P is added to θ.* We have therefore

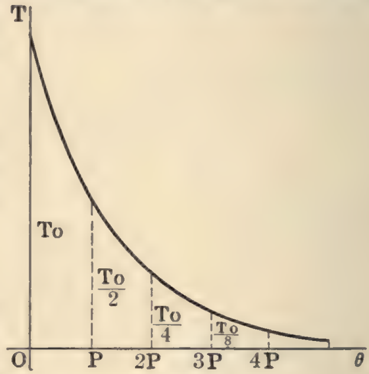


FIG. 45.

$$\tfrac{1}{2}T = T_0e^{-\mu(\theta+P)}.$$

* The difference between this definition of P and the one given in the preceding section is accounted for by the difference in the signs of the exponents in equation (4) and in equation (14) of the preceding section.

Dividing equation (4) by the last equation we get

$$\left.\begin{aligned} 2 &= e^{\mu P}, \\ \text{or} \qquad P &= \frac{1}{\mu}\log_e 2 \\ &\doteq \frac{.7}{\mu}. \end{aligned}\right\} \tag{5}$$

Thus if $\theta = nP$, then by equations (4) and (5)

$$T = \frac{T_0}{2^n}. \tag{6}$$

Therefore taking 0.53 for hemp rope on oak and $\theta = 2\pi$, we obtain $n = 4.76$ and $2^n = 27.3$. Hence in this case T_0 is 27.3 times as great as T.

Application to Belts. — The tensile force on one side of a belt which transmits power is greater than that on the other side. The relation between the tensile forces on the two sides of the belt is given by equation (4). Thus if T_1 denotes the tensile force on the driving side and T_2 that on the slack side, then

$$T_2 = T_1 e^{-\mu\theta} \quad \text{or} \quad T_1 = T_2 e^{\mu\theta}. \tag{4'}$$

The difference between T_1 and T_2 is the effective force which drives the pulley. Denoting the effective force by F, we have

$$\left.\begin{aligned} F &= T_1 - T_2 \\ &= T_1 (1 - e^{-\mu\theta}) \\ &= T_2 (e^{\mu\theta} - 1). \end{aligned}\right\} \tag{7}$$

We have neglected the cross-section of the cord in the solution of the foregoing problem. Therefore the results which we have obtained are applicable to actual problems only when the cross-section of the cord is negligible compared with that of the solid with which it is in contact.

PROBLEMS.

1. A weight of 5 tons is to be raised from the hold of a ship by means of a rope which takes $3\frac{1}{2}$ turns around the drum of a steam windlass. If $\mu = 0.25$ what force must a man exert at the other end of the rope?

2. By pulling with a force of 200 pounds a man just keeps from surging a rope, which takes 2.5 turns around a post. Find the tensile force at the other end of the rope. $\mu = 0.2$.

3. A weight W is suspended by a rope which makes $1\frac{1}{4}$ turns around a clamped pulley and goes to the hand of a workman. If $\mu = 0.2$, find the force the man has to apply in order (a) to support the weight, (b) to raise it.

4. Two men, each of whom can exert a pull of 250 pounds, can support a weight by means of a rope which takes 2 turns around a post. On the other hand, one of the men can support it alone if the rope makes 2.5 turns. Find the weight.

5. In order to prevent surging a sailor has to exert a force of 150 pounds at the end of a hawser, which is used to keep the stern of a boat at rest while the bow is being turned by the engines. Find the pull exerted by the boat upon the hawser under the following conditions:

[*Hint.—Make use of equations* (5) *and* (6).]

(a) $\theta = \frac{\pi}{4}$, $\mu = 0.2$. (g) $\theta = 2\pi$, $\mu = 0.1$.

(b) $\theta = \frac{\pi}{4}$, $\mu = 0.5$. (h) $\theta = \frac{9\pi}{4}$, $\mu = 0.4$.

(c) $\theta = \frac{\pi}{2}$, $\mu = 0.5$. (i) $\theta = \frac{5\pi}{2}$, $\mu = 0.5$.

(d) $\theta = \pi$, $\mu = 0.4$. (j) $\theta = 3\pi$, $\mu = 0.3$.

(e) $\theta = \frac{5\pi}{4}$, $\mu = 0.3$. (k) $\theta = \frac{13\pi}{4}$, $\mu = 0.4$.

(f) $\theta = \frac{3\pi}{2}$, $\mu = 0.2$. (l) $\theta = \frac{7\pi}{2}$, $\mu = 0.5$.

6. A belt has to transmit an effective force of 500 pounds. Find the tensile force on both sides of the belt, under the following conditions:

(a) $\theta = 135°$, $\mu = 0.5$. (e) $\theta = 165°$, $\mu = 0.2$.
(b) $\theta = 135°$, $\mu = 0.4$. (f) $\theta = 180°$, $\mu = 0.3$.
(c) $\theta = 150°$, $\mu = 0.3$. (g) $\theta = 180°$, $\mu = 0.5$.
(d) $\theta = 165°$, $\mu = 0.5$. (h) $\theta = 195°$, $\mu = 0.4$.

7. In the preceding problem find the width of the belt, supposing the permissible safe tensile force to be 50 pounds per inch of its width.

CHAPTER V.

MOTION.

FUNDAMENTAL MAGNITUDES.

66. Analysis of Motion.—The conception of motion necessarily involves four ideas, namely, the ideas of

(a) A body which moves.

(b) A second body with respect to which it moves.

(c) A distance which it covers.

(d) An interval of time during which the distance is covered.

67. Relativity of Motion. Reference System. — The first important inference to be drawn from the foregoing analysis is the fact that motion presupposes at least two bodies, namely, the body which is supposed to move and the body to which the motion is referred. The words "motion" and "rest" become meaningless when applied to a single particle with no other body for reference. Whenever we think or talk about the motion of a particle we refer its motion, consciously or unconsciously, to other bodies. The body to which motion is referred is called a *reference system.* The choice of a particular body as a reference system is a question of convenience. If a man walks in a crowded car fast enough to discommode its occupants he will be blamed, not because he is moving at the rate of, say, 20 miles per hour with respect to the ground, but because he is moving at the rate of 4 miles per hour with respect to the car. In this case the car should be taken as the reference system, and not the ground. On the other hand if the man wants to leave the moving car, it is of great importance for him to

consider the velocity with which he is going to land. In this case, therefore, the surface of the earth should be taken as the reference system.

68. Fundamental Magnitudes. — The first two of the four conceptions into which we analyzed motion are similar; therefore three distinct conceptions are associated with motion. The first of these is the idea of body, or of matter; the second is that of distance, and the third is that of time. Distance and time are terms which are too familiar to be made clearer by definitions, therefore we will not attempt to define them.

In their efforts to reduce natural phenomena to their simplest terms scientists have come to the conclusion that all physical phenomena are the result of motion. It is the main object of science to describe the complicated phenomena of nature in terms of motion, in other words, to express all physical magnitudes in terms of the three magnitudes involved in motion. Therefore time, mass, and length are called *fundamental magnitudes* and all others *derived magnitudes.*

69. Fundamental Units.—The units of time, length, and mass are called *fundamental units,* while those of other magnitudes are called *derived units.*

70. The Unit of Time is $\frac{1}{86,400}$ part of the mean solar day, and is called the *second.*

71. The Unit of Length is the *centimeter,* which is $\frac{1}{100}$ part of the standard *meter.* The latter is the distance at 0° C. between two parallel lines drawn upon a certain platinum-iridium bar in the possession of the French government.

72. Mass.—The choice of the units of time and length is comparatively easy. We associate only one property with each of these quantities, therefore in choosing a unit all we have to do is to decide upon its size. Matter, on the other hand, has a great number of properties, such as volume, shape, temperature, weight, mass, elasticity, etc. We com-

pare and identify different bodies by means of these properties. In selecting one of these properties to represent the body in our study of motion we must see that the property fulfills two conditions: that it is intimately related to motion and that it is constant.

Weight is often used to represent a body in its motion. So far as bodies on the earth are concerned weight is intimately connected with motion, but it is not constant. Besides, when bodies are far from the earth, weight does not have a definite meaning. Therefore weight does not satisfy the foregoing conditions. The property which serves the purpose best is known as *mass.* It is intimately connected with motion and is constant.* The nature of this property will be discussed in the next chapter. Therefore we will content ourselves by defining mass as *that property with which bodies are represented in discussions of their motion.*

73. Unit of Mass.—The unit of mass is the *gram*, which is $\frac{1}{1000}$ part of the mass of the standard *kilogram.* The latter is the mass of a piece of platinum in the possession of the French Government.

74. Dimensions.—The fundamental magnitudes enter into the composition of one derived magnitude in a manner different from the way they enter into that of a second. Length alone enters into the composition of an area, while velocity contains both length and time, and all three of the fundamental magnitudes combine in work and momentum. The expression which gives the manner in which time, length, and mass combine to form a derived magnitude is called the *dimensional formula* of that magnitude. Thus the dimensional formulæ for area, velocity, and momentum are, respectively,

$$[A] = [L^2], \quad [V] = [LT^{-1}], \text{ and } [H] = [MLT^{-1}],$$

where M, L, and T represent length, mass, and time. The exponent of each letter is called the dimension of the de-

* Cf. § 101.

rived magnitude in the fundamental magnitude which the letter represents. Thus area has two dimensions in length and zero dimension in both time and mass, while momentum has one dimension in mass, one dimension in length, and minus one dimension in time.

75. Homogeneity of Equations. — Magnitudes of different dimensions can neither be added nor subtracted. Therefore in a true equation the sum of the magnitudes of one kind which are on the left of the equation sign equals the sum of the magnitudes of the same kind which are on the right. When all the terms of an equation have the same dimensions the equation is said to be *homogeneous*.

76. Systems of Units. — The C.G.S. System is used in most of the civilized countries and by scientists all over the world. In this system the centimeter, the gram, and the second are the *fundamental units*.

English-speaking people use another system, known as the British gravitational system, in which weight, length, and time are the fundamental magnitudes and the pound, the foot, and the second are the fundamental units. Thus the unit of time is the same in both systems. The following equations give the relation between the centimeter and the inch with an error of less than one-tenth of one per cent.

$$\begin{aligned} 1 \text{ in.} &= 2.54 \text{ cms.} \\ 1 \text{ cm.} &= 0.3937 \text{ in.} \end{aligned}$$

The relation between the mass of a body which weighs one pound and the gram is given by the following equations with an error of less than one-tenth of one per cent.

$$\begin{aligned} 1 \text{ kg.} &= 2.205 \text{ pds.} \\ 1 \text{ pd.} &= 453.6 \text{ gms.,} \end{aligned}$$

where kg. is the abbreviation for the *kilogram,* or 1000 gms., while pd. denotes the mass of a body which weighs one pound in London and is often called *pound-mass.* Denoting

the pound (weight) by its usual abbreviation we have

$$2.205 \text{ lbs.} = \text{the weight of } 1000 \text{ gms.*}$$

VELOCITY.

77. Displacement. — When the position of a particle with respect to a reference system is slightly changed it is said to have been *displaced*, and the vector **s**, Fig. 46, which has its origin at the initial position and its terminus at the final position, is called a *displacement*.

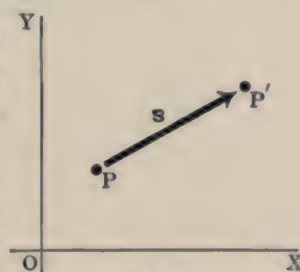


Fig. 46.

78. Velocity. — If a particle undergoes equal displacements in equal intervals of time, however small these intervals, it is said to have a *constant velocity*. In this particular case the velocity equals, numerically, the distance covered per second. When, therefore, a distance s is covered in an interval of time t, the velocity is given by

$$v = \frac{s}{t}.$$

By equal displacements are meant displacements equal in magnitude and the same in direction. Therefore constant velocity means a velocity which is constant in direction as well as in magnitude. The magnitude of velocity without regard to its direction is called *speed*.

In general, bodies not only cover unequal distances in equal intervals of time, but also change their directions of motion. Therefore we need a definition of velocity like the following, which is perfectly general.

The velocity of a particle at any point of its path equals, in magnitude, the time rate at which it describes that part of the path which is in the immediate neighborhood of the point and has the direction of the tangent at that point.

* For the relation between mass and weight see p. 109.

In order to express this definition of velocity in analytical language, consider a particle describing a curved path with a changing speed. The most natural way of determining the speed at a point P, Fig. 47, is to observe the interval of time which the particle takes to pass two points, P_1 and P_2, which are equidistant from P, then to divide the distance P_1P_2 by that interval of time. This gives the average speed from P_1 to P_2, which may or may not equal the actual speed at P. If, however, we take the points P_1 and P_2 nearer to P we obtain an average speed which is, in general, nearer the speed at P, because there is less chance for large variations. If we take P_1 and P_2 nearer and nearer the average speed approaches more and more to the value at P. Therefore the limiting value of the ratio $\frac{P_1P_2}{t}$ is the speed at P. In other words

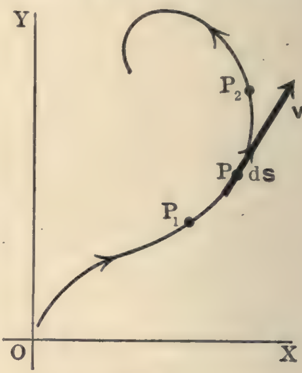


FIG. 47.

$$v = \frac{ds}{dt} = \dot{s}\,^* \qquad \text{(I)}$$

is the analytical definition of speed. Therefore the velocity is a vector which has $\dot{s}$ for its magnitude and which is tangent to the path at the point considered, that is,

$$\mathbf{v} = \dot{\mathbf{s}}. \qquad \text{(I}'\text{)}$$

* The Differential Calculus was invented by Newton and Leibnitz independently. Newton adopted a notation in which the derivative of a variable s with respect to another variable is denoted by $\dot{s}$. This notation is not convenient when derivatives are taken with respect to several variables. The notation introduced by Leibnitz is more convenient and is the notation which is generally adopted. Newton's notation, however, is often used to denote differentiation with respect to time. On account of the compactness of $\dot{s}$ compared with $\frac{ds}{dt}$, we will denote differentiations with respect to time by Newton's notation whenever compactness of expression is desired.

79. Dimensions and Units of Velocity. — The dimensions of velocity are $[LT^{-1}]$. The C.G.S. unit of velocity is the *centimeter per second*, $\frac{cm.}{sec.}$. The British unit of velocity is the *foot per second*, $\frac{ft.}{sec.}$.

80. Rectangular Components of Velocity. — Let $\mathbf{v}$, Fig. 48,

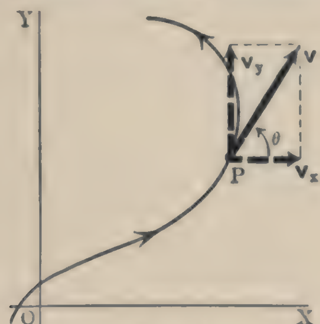


Fig. 48.

denote the velocity at P, then the magnitude of its component along the x-axis is

$$\left.\begin{aligned} v_x &= v\cos\theta \\ &= \frac{ds}{dt}\cos\theta \\ &= \frac{ds\cos\theta}{dt} \\ &= \frac{dx}{dt} = \dot{x}. \\ \text{Similarly}\qquad v_y &= \frac{dy}{dt} = \dot{y}, \\ \text{and}\qquad v_z &= \frac{dz}{dt} = \dot{z}. \end{aligned}\right\} \qquad \text{(II)}$$

Equations (II) state that the component of the velocity of a particle along any line equals the velocity of the projection of the particle upon that line, in other words, the ve-

locity along any direction equals the rate at which distance is covered along that direction.

The velocity and its components evidently fulfill the relation

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}. \qquad \text{(III)}$$

When, as in the case of Fig. 48, the particle moves in the xy-plane, $\dot{z} = 0$, therefore

$$v = \sqrt{v_x^2 + v_y^2}. \qquad \text{(III')}$$

The direction of **v**, in this case, is given by

$$\tan\theta = \frac{\dot{y}}{\dot{x}}, \qquad \text{(IV)}$$

where θ is the angle **v** makes with the x-axis.

ILLUSTRATIVE EXAMPLE.

Find the path, the velocity, and the components of the velocity of a particle which moves so that its position at any instant is given by the following equations:

$$x = at, \qquad \text{(a)}$$
$$y = -\tfrac{1}{2}gt^2. \qquad \text{(b)}$$

Eliminating t between (a) and (b), we obtain

$$x^2 = -\frac{2a^2}{g}y;$$

for the equation of the path, therefore the path is a parabola, Fig. 49.

To find the component-velocities we differentiate (a) and (b) with respect to the time. This gives

$$\dot{x} = a,$$
$$\dot{y} = -gt.$$
$$\therefore \quad v = \sqrt{a^2 + gt^2}.$$

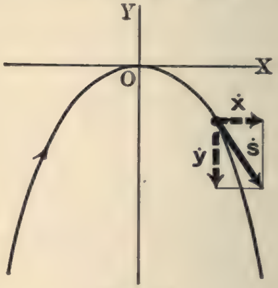


Fig. 49.

Discussion. — The horizontal component of the velocity is directed to the right and is constant, while the vertical component is directed downwards and increases at a constant rate.

We will see later that these equations represent the motion of a body which is projected horizontally from an elevated position.

PROBLEMS.

1. Find the path and the velocity of a particle which moves so that its position at any instant is given by the following pairs of equations:

(a) $x = at$, $y = bt$.
(b) $x = at$, $y = at - \frac{1}{2} gt^2$.
(c) $x = at$, $y = b \cos \omega t$.
(d) $x = a \sin \omega t$, $y = bt$.
(e) $x = a \sin \omega t$, $y = a \cos \omega t$.
(f) $x = a \sin \omega t$, $y = b \sin \omega t$.
(g) $x = ae^{kt}$, $y = ae^{-kt}$.

2. Prove the relation $v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$.

81. Radial and Transverse Components of Velocity. — The magnitude of the velocity along the radius vector is, according to the results of the preceding section,

$$v_r = \frac{dr}{dt} = \dot{r}. \tag{1}$$

The expression for the velocity at right angles to r is obtained by considering the motion of the projection of the particle along a perpendicular to r. When the particle moves through ds, its projection moves through $r\,d\theta$, Fig. 50, therefore the required velocity is

$$\left.\begin{aligned} v_p &= \frac{r\,d\theta}{dt} \\ &= r\frac{d\theta}{dt} = r\dot{\theta}. \end{aligned}\right\} \tag{2}$$

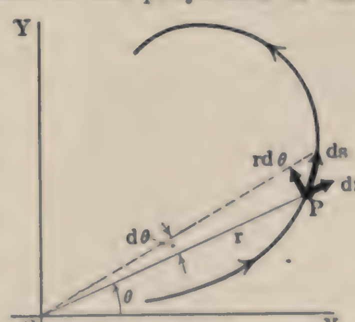


Fig. 50.

The components v_r and v_p may be expressed in terms of $\dot{x}$ and $\dot{y}$ by differentiating the equations of transformation

$$r^2 = x^2 + y^2 \tag{3}$$

and

$$\theta = \tan^{-1}\frac{y}{x} \tag{4}$$

with respect to the time. Differentiating (3) we obtain

$$\begin{aligned} v_r &= \frac{dr}{dt} \\ &= \frac{x}{r}\frac{dx}{dt} + \frac{y}{r}\frac{dy}{dt} \\ &= \dot{x}\cos\theta + \dot{y}\sin\theta. \end{aligned} \tag{5}$$

Differentiating (4) we get

$$\begin{aligned} v_p &= r\frac{d\theta}{dt} \\ &= r\frac{x\dot{y} - y\dot{x}}{x^2 + y^2} \\ &= \dot{y}\cos\theta - \dot{x}\sin\theta. \end{aligned} \tag{6}$$

These components satisfy the relation

$$v = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}. \tag{7}$$

ILLUSTRATIVE EXAMPLE.

A particle describes the motion defined by the equations

$$x = a\cos kt, \tag{a}$$

and

$$y = a\sin kt. \tag{b}$$

Find the equation of the path, the velocity at any instant, and the components of the latter.

Squaring and adding (a) and (b) we eliminate t and obtain

$$x^2 + y^2 = a^2$$

for the equation of the path.

Differentiating (a), we have

$$\begin{aligned} \dot{x} &= \frac{dx}{dt} \\ &= -ka\sin kt \\ &= -ky. \end{aligned}$$

Differentiating (b), we obtain

$$\begin{aligned} \dot{y} &= \frac{dy}{dt} \\ &= ka\cos kt \\ &= kx. \end{aligned}$$

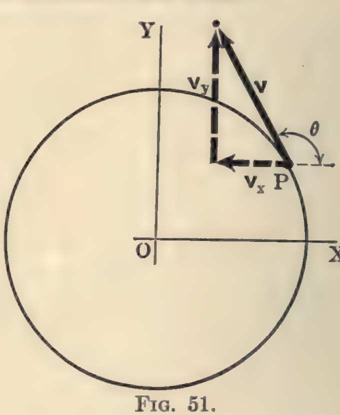


Fig. 51.

Therefore
$$\begin{aligned} v &= \sqrt{\dot{x}^2 + \dot{y}^2} \\ &= k\sqrt{x^2 + y^2} \\ &= ka. \end{aligned}$$

Thus the particle describes a circle with a constant speed ka. The direction of the velocity at any instant is given by the relation

$$\begin{aligned} \tan\theta &= \frac{\dot{y}}{\dot{x}} \\ &= -\frac{y}{x}. \end{aligned}$$

The components v_r and v_p may be obtained at once by remembering, (1) that the radius vector is constant: e.g., $\dot{r} = 0$, (2) that it is always normal to the path: e.g., $rd\theta = ds$. Therefore

$$v_r = \frac{dr}{dt} = 0,$$

and
$$v_p = r\frac{d\theta}{dt} = \frac{ds}{dt} = v = ka.$$

82. Velocity of a Particle Relative to Another Particle in Motion.—Consider the motion of a particle P_1, Fig. 52, with

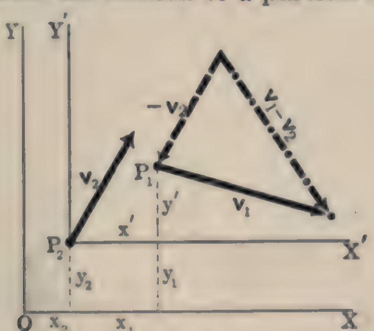


Fig. 52.

respect to a particle P_2, when both are in motion relative to the system of axes XOY.

Let the system of axes $X'P_2Y'$ have P_2 for its origin and move with its axes parallel to those of the system XOY. Further let (x_1, y_1) and (x_2, y_2) be the positions, and $\mathbf{v}_1$ and $\mathbf{v}_2$ the velocities of P_1 and P_2 with respect to XOY. Then

if (x', y') denotes the position and $\mathbf{v}'$ the velocity of P_1 with respect to $X'P_2Y'$, we get

$$x' = x_1 - x_2,$$
$$y' = y_1 - y_2.$$

Differentiating the last two equations with respect to the time

$$\dot{x}' = \dot{x}_1 - \dot{x}_2,$$
$$\dot{y}' = \dot{y}_1 - \dot{y}_2.$$

Therefore
$$\begin{aligned}\mathbf{v}' &= \dot{\mathbf{x}}' + \dot{\mathbf{y}}' \\ &= (\dot{\mathbf{x}}_1 + \dot{\mathbf{y}}_1) - (\dot{\mathbf{x}}_2 + \dot{\mathbf{y}}_2) \\ &= \mathbf{v}_1 - \mathbf{v}_2. \qquad \text{(V)}\end{aligned}$$

Equation (V) states that the velocity of a particle with respect to another particle is obtained by subtracting the velocity of the first from that of the second.

ILLUSTRATIVE EXAMPLE.

Two particles move in the circumference of a circle with constant speeds of v and $2\,v$. Find their relative velocities.

Let the slower one be chosen as the reference particle, and let the angle P_2OP_1, Fig. 53, be denoted by θ. Then the velocity of P_1 relative to P_2 is

$$\mathbf{v}_1' = \mathbf{v}_1 - \mathbf{v}_2.$$

But $v_1 = 2\,v$ and $v_2 = v$, therefore

$$\begin{aligned}v' &= \sqrt{4\,v^2 - 2\,v \cdot 2\,v \cdot \cos\theta + v^2} \\ &= v\sqrt{5 - 4\cos\theta}.\end{aligned}$$

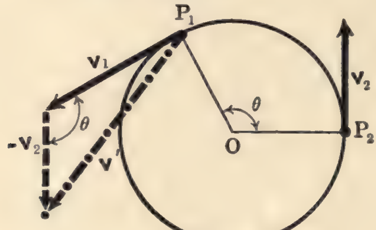


Fig. 53.

Discussion. — Whenever P_1 passes P_2 the value of θ is a multiple of $2\,\pi$, therefore $\cos\theta = 1$ and $v' = v$. When the particles occupy the ends

of a diameter $\cos\theta = -1$, therefore $v_1' = 3v$. When they are separated by an angle which is an odd multiple of $\frac{\pi}{2}$, $\cos\theta = 0$; therefore $v_1 = v\sqrt{5}$.

PROBLEMS.

1. An automobile is moving at the rate of 30 miles an hour in a direction at right angles to a train which is making 40 miles an hour; find the velocity of the automobile with respect to the train.

2. Two trains pass each other on parallel tracks, in opposite directions. A passenger in one of the trains observes that it takes the other train 4 seconds to pass him. What is the length of the other train if the velocities of the two trains are 50 and 40 miles per hour?

3. A man of height h walks on a level street away from an electric lamp of height H. If the velocity of the man is v, find the velocity of the end of his shadow (a) with respect to the ground and (b) with respect to the man.

4. Two particles move, in opposite directions, on the circumference of the same circle with the same constant speed. Find an expression for their relative velocity and see what this expression becomes at special positions of the particles.

5. A train is moving due north at the rate of 50 miles an hour. The wind is blowing from the southeast with a velocity of 20 miles an hour. Find the apparent direction and magnitude of the wind to a man on the train.

6. The wind seems to blow from the north to an automobile party traveling westward at the rate of 15 miles an hour. On doubling the speed of the automobile the wind appears to come from the northwest. Find the actual direction and magnitude of the velocity of the wind.

7. Find the velocity of a particle moving on the circumference of a circle with uniform speed relative to another particle moving with equal speed in a diameter of the circle.

8. Express the speed of a mile a minute in the C.G.S. units.

9. Express the C.G.S. unit of velocity in miles per hour.

10. Prove that $\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2$.

11. Prove analytically that

$$v_x = v_r\cos\theta - v_p\sin\theta,$$
$$v_y = v_r\sin\theta + v_p\cos\theta.$$

12. Prove graphically that

$$v_r = v_x\cos\theta + v_y\sin\theta,$$
$$v_p = v_y\cos\theta - v_x\sin\theta.$$

ANGULAR VELOCITY.

83. Angular Displacement.—When the motion of a particle is referred to an axis, then the angle which the *axial plane*, i.e., the plane determined by the particle and the axis, describes, is called an *angular displacement*. Angular displacement is a vector magnitude which is represented by a vector drawn along the axis; as in the case of the vector representation of a torque. The directional relations are the same; that is, the vector points towards the observer and is considered as positive when the rotation is counter-clockwise. It points away from the observer and is negative when the rotation is clockwise.

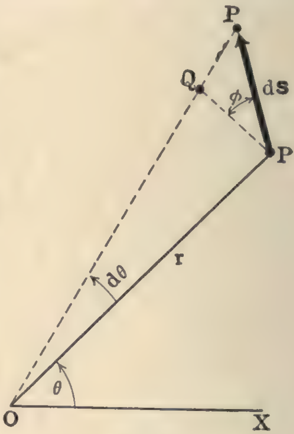


FIG. 54.

The relation between the linear displacement of a particle and its angular displacement about an axis may be found from a consideration of Fig. 54:

$$d\theta = \frac{PQ}{r}$$

$$= \frac{ds \cos \phi}{r},$$

where ds is the linear displacement of the particle P, $d\theta$ is the corresponding displacement about an axis through the point O perpendicular to the plane of the paper, and ϕ is the angle ds makes with the normal to the axial plane.

When r is constant ϕ is zero, and the particle describes a circle, in which case the last equation becomes

$$d\theta = \frac{ds}{r} \quad \text{or} \quad \theta = \frac{s}{r}.$$

84. Unit Angle. — In the last equation $\theta = 1$ when $s = r$; therefore the angle which is subtended at the center of a circle by an arc equal to the radius is the unit of angle. This unit is called the *radian*. Angles and angular displacements have no dimensions. Why?

85. Angular Velocity. — The conception of angular velocity is similar to that of linear velocity. It is the time rate at which the axial plane sweeps over an angle. When constant it is numerically equal to the angle swept over per second. If we denote the angular velocity by $\boldsymbol{\omega}$ its magnitude is defined by

$$\omega = \frac{d\theta}{dt} = \dot{\theta}. \tag{VI}$$

Angular velocity is a vector quantity which is represented by a vector drawn along the axis of rotation. The vector points towards the observer when the rotation is counterclockwise, and away from the observer when it is clockwise. The angular velocity is said to be positive in the first case and negative in the second case. Angular velocity has the dimensions of the reciprocal of time.

$$[\Omega] = [T^{-1}].$$

The unit of angular velocity is the *radian per second*, $\frac{rad.}{sec.}$.

The relation between the linear and the angular velocities of a particle may be obtained from equations (VI) and (I).

$$\begin{aligned} \omega &= \frac{d\theta}{dt} \\ &= \frac{\frac{ds \cos\phi}{r}}{dt} \\ &= \frac{v \cos\phi}{r} \\ &= \frac{v_x}{r}, \end{aligned} \tag{VII}$$

where v_p is the component of the linear velocity in a direction perpendicular to the axial plane.

ILLUSTRATIVE EXAMPLE.

A particle describes a circle of radius a with a constant speed v. Find its angular velocity relative to an axis through a point on the circumference and perpendicular to the plane of the circle.

Let P (Fig. 55) be the position of the particle and O the point at which the axis of reference intersects the circle. Move the particle from P to P' and denote the linear and angular displacements by ds and $d\theta$ respectively. Then the angle subtended by PP' at O is one-half that subtended at C. Hence

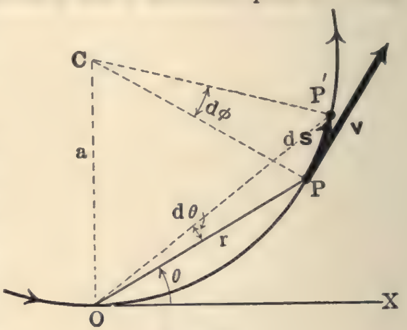


FIG. 55.

$$d\theta = \frac{1}{2} d\phi$$

$$= \frac{1}{2}\frac{ds}{a}.$$

$$\therefore \qquad \omega = \frac{d\theta}{dt}$$

$$= \frac{1}{2a}\frac{ds}{dt}$$

$$= \frac{v}{2a}.$$

Thus the angular velocity about O is independent of the position of the particle and equals one-half the angular velocity about the center.

PROBLEMS.

1. The radius of the earth is 4000 miles and that of its orbit 93 million miles. Compare the angular velocities of a point on the equator with respect to the sun at midday and midnight.

2. In what latitude is a bullet, which is projected east with a velocity of 1320 feet per second, at rest relatively to the earth's axis; the radius being taken as 4000 miles?

3. A belt passes over a pulley which has a diameter of 30 inches and which makes 200 revolutions per minute. Find the linear speed of the belt and the angular speed of the pulley.

4. The wheels of a bicycle, which are 75 cm. in diameter, make 5000 revolutions in 65 minutes. Find the speed of the rider; the angular speed of the wheels about their axles; the relative velocity of the highest point of each wheel with respect to the center.

5. A point moves with a constant velocity $\mathbf{v}$. Find its angular velocity about a fixed point whose distance from the path is a.

6. A railroad runs due west in latitude λ. Find the velocity of the train if it always keeps the sun directly south of it.

7. Find the expression for the angular velocity of any point on the rim of a wheel of radius a, moving with a velocity v; the wheel is supposed to be rolling without slipping. Discuss the values of the velocity for special points.

8. In the preceding problem find the relative velocity of any point on the rim with respect to the center of the wheel, and the velocity of the center with respect to the point of contact with the ground.

9. The end of a vector describes a circle at a constant rate. If the origin is outside the circle find the velocity along and at right angles to the vector. Discuss the values for interesting special positions.

10. In the preceding problem derive an expression for the angular velocity of the vector and discuss it.

ACCELERATION.

86. Acceleration.—When the velocity of a particle changes it is said to have an *acceleration*. The change may be in the magnitude of the velocity, in the direction, or in both; further it may be positive or negative. Therefore the term *acceleration* includes retardation as well as increase in velocity. Retardation is negative acceleration.

If the particle moves in a straight path with a velocity which increases or diminishes at a constant rate its acceleration equals, numerically, the change in the velocity per second and is said to be constant:

$$\mathbf{f} = \frac{\mathbf{v}_2 - \mathbf{v}_1}{t},$$

where $\mathbf{f}$ is the acceleration and $\mathbf{v}_1$ and $\mathbf{v}_2$ are the velocities at the beginning and at the end of the interval of time t.

Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are in the same line, their difference will be a vector in the same direction. Therefore in this particular case the acceleration is constant not only in magnitude but also in direction.

The following definition of acceleration is general and holds true whatever the manner in which the velocity changes.

The magnitude of the acceleration of a particle at any point of its path equals the time rate at which its velocity changes at the instant it occupies that point.

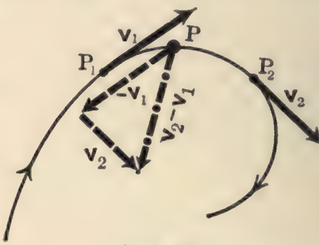


FIG. 56.

The analytical expression for this definition may be obtained by a reasoning similar to that employed in deriving the analytical definition of velocity. Suppose it is required to find the acceleration at P (Fig. 56). Let $\mathbf{v}_1$ and $\mathbf{v}_2$ denote the velocities at two neighboring points P_1 and P_2. Then the ratio

$$\bar{\mathbf{f}} = \frac{\mathbf{v}_2 - \mathbf{v}_1}{t}$$

gives the average rate at which the velocity changes during the interval of time t, which it takes the particle to move from P_1 to P_2. Therefore $\bar{\mathbf{f}}$ is the average acceleration for that interval of time. In general this average acceleration will not be the same as the acceleration at P. But by taking P_1 and P_2 nearer and nearer to P the difference between the average acceleration and the required acceleration may be made as small as desired. Therefore at the limit when P_1, P, and P_2 become successive positions of the particle, the average acceleration becomes identical with the acceleration at P, and the last equation takes the form

$$\mathbf{f} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}}. \tag{VIII}$$

It must be remembered that $\mathbf{dv}$ is the vector difference of the velocities at the beginning and at the end of the interval of time dt; therefore $\mathbf{f}$ is a vector magnitude with a direction which is, in general, different from that of the velocity.

87. Dimensions and Units of Acceleration. — The dimensions of acceleration are $[LT^{-2}]$. The unit of acceleration is a unit change in the velocity per second. Therefore the C.G.S. unit is $\frac{\text{cm./sec.}}{\text{sec.}}$ or $\frac{\text{cm.}}{\text{sec.}^2}$. Thus if the velocity of a particle increases by an amount of one $\frac{\text{cm.}}{\text{sec.}}$ during each second it has a unit acceleration. The engineering unit of acceleration is the foot per sec. per sec., $\frac{\text{ft.}}{\text{sec.}^2}$.

PROBLEMS.

1. Express the engineering unit of acceleration in terms of the C.G.S. unit.

2. Taking the value of the gravitational acceleration to be 980 $\frac{\text{cm.}}{\text{sec.}^2}$, find its value in $\frac{\text{ft.}}{\text{sec.}^2}$ and $\frac{\text{miles}}{\text{hr.}^2}$.

3. A train moving at the rate of 30 kilometers per hour is brought to rest in two minutes. Find the average acceleration and express it in terms of $\frac{\text{cm.}}{\text{sec.}^2}$, $\frac{\text{ft.}}{\text{sec.}^2}$ and $\frac{\text{km.}}{\text{hr.}^2}$.

88. Components of Acceleration along Rectangular Axes. — Suppose a particle to describe a path in the xy-plane. Then if $\mathbf{v}_1$ and $\mathbf{v}_2$ be the velocities at two neighboring points, we can write

$$\begin{aligned} \mathbf{dv} &= \mathbf{v}_2 - \mathbf{v}_1 \\ &= (\dot{\mathbf{x}}_2 + \dot{\mathbf{y}}_2) - (\dot{\mathbf{x}}_1 + \dot{\mathbf{y}}_1) \\ &= \mathbf{d}\dot{\mathbf{x}} + \mathbf{d}\dot{\mathbf{y}}. \end{aligned}$$

$$\therefore \quad \mathbf{f} = \frac{\mathbf{dv}}{dt} = \frac{\mathbf{d}\dot{\mathbf{x}}}{dt} + \frac{\mathbf{d}\dot{\mathbf{y}}}{dt}.$$

But since

$$\mathbf{f} = \mathbf{f}_x + \mathbf{f}_y,$$

$$\mathbf{f}_x + \mathbf{f}_y = \frac{\mathbf{d}\dot{\mathbf{x}}}{dt} + \frac{\mathbf{d}\dot{\mathbf{y}}}{dt}.$$

The last equation cannot be true unless

$$\mathbf{f}_x = \frac{d\dot{\mathbf{x}}}{dt}$$

and

$$\mathbf{f}_y = \frac{d\dot{\mathbf{y}}}{dt}.$$

Therefore the component of the acceleration along a fixed line equals the time rate of change of the component of the velocity along that line.

It follows from the last two equations that:

$$\left.\begin{aligned} f_x &= \frac{d\dot{x}}{dt} = \frac{d^2x}{dt} = \ddot{x}, \\ f_y &= \frac{d\dot{y}}{dt} = \frac{d^2y}{dt} = \ddot{y}. \end{aligned}\right\} \tag{IX}$$

The magnitude and the direction of the acceleration are given by the following equations:

$$f = \sqrt{\ddot{x}^2 + \ddot{y}^2}, \tag{X}$$

$$\tan\theta = \frac{\ddot{y}}{\ddot{x}}, \tag{XI}$$

where θ is the angle $\mathbf{f}$ makes with the x-axis.

89. Tangential and Normal Components of Acceleration. — The tangential component of the acceleration at P (Fig. 57) equals the rate at which the velocity increases along the direction of the tangent at P. In order to find this rate we consider the velocities at two neighboring points P_1 and P_2. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the velocities at these points and ϵ_1 and ϵ_2 the angles which $\mathbf{v}_1$ and $\mathbf{v}_2$ make with the tangent at P. Then the change in the velocity along the tangent at P, while the particle moves from P_1 to P_2, is

$$v_2 \cos \epsilon_2 - v_1 \cos \epsilon_1.$$

Dividing this by the corresponding interval of time we obtain the average rate at which the velocity increases from

P_1 to P_2 along the tangent at P. Therefore the average tangential acceleration is

$$\bar{f}_\tau = \frac{v_2 \cos \epsilon_2 - v_1 \cos \epsilon_1}{t}.$$

This average approaches the actual tangential acceleration at P as P_1 and P_2 are made to approach P as a limit. But

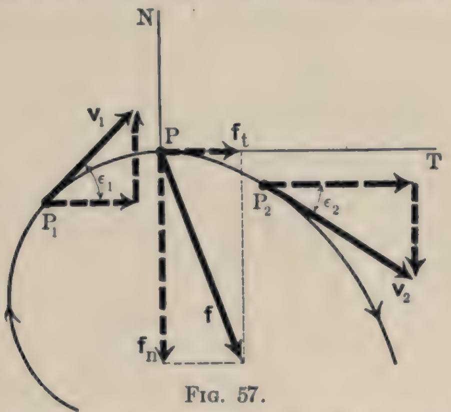


FIG. 57.

as these points approach P the angles ϵ_1 and ϵ_2 approach zero as a limit and their cosines approach unity. Therefore the tangential acceleration at P is

$$\begin{aligned} f_\tau &= \underset{t \doteq 0}{\text{limit}} \left[\frac{v_2 - v_1}{t}\right] \\ &= \frac{dv}{dt} = \dot{v} \\ &= \frac{d^2s}{dt^2} = \ddot{s}. \end{aligned}$$

By similar reasoning we obtain

$$\bar{f}_n = \frac{-v_2 \sin \epsilon_2 - v_1 \sin \epsilon_1}{t},$$

for the average normal acceleration between P_1 and P_2. The actual normal acceleration at P is the limiting value of this expression as P_1 and P_2 approach P. But as these points approach P, v_1 and v_2 approach v, the velocity at P, while

$\sin \epsilon_1$ and $\sin \epsilon_2$ approach ϵ_1 and ϵ_2,* respectively. Therefore the normal acceleration at P is

$$f_n = \underset{t \doteq 0}{\text{limit}} \left[-v \frac{\epsilon_1 + \epsilon_2}{t} \right]$$

$$= \underset{t \doteq 0}{\text{limit}} \left[-v \frac{\theta}{t} \right]$$

$$= -v \frac{d\theta}{dt} = -v\dot{\theta}, \qquad \text{(XII)}$$

where $\theta = \epsilon_1 + \epsilon_2$ is the total change in the direction of the velocity in going from P_1 to P_2. Since the direction of the velocity coincides with that of the tangent, $\dot{\theta}$ is the rate at which the directions of the tangent and the normal change. But the rate at which the normal changes its direction equals the angular velocity of the particle about the center of curvature. Therefore if ρ denotes the radius of curvature at P, we have

$$\frac{d\theta}{dt} = \frac{v}{\rho}, \qquad \text{[by VII]}$$

and

$$f_n = -\frac{v^2}{\rho}. \qquad \text{(XIII)}$$

The negative sign in (XIII) shows that f_n and ρ are measured in opposite directions. Since ρ is measured from the center of curvature, f_n must be directed towards the center of curvature. Therefore the total acceleration is always directed towards the concave side of the path.

The following are the principal results obtained in this section and the conclusions to be drawn from them.

(*a*) The magnitude of the tangential acceleration is $\dot{v}$;

$$f_r = \dot{v}.$$

(*b*) The normal acceleration is directed towards the center of curvature and has $\frac{v^2}{\rho}$ for its magnitude;

$$f_n = -\frac{v^2}{\rho}.$$

* See Appendix Avi.

(*c*) The magnitude of the total acceleration is given by the relation

$$f = \sqrt{\dot{v}^2 + \frac{v^4}{\rho^2}}.$$

(*d*) The total acceleration is directed towards the concave side of the path and makes an angle with the tangent which is defined by

$$\tan \psi = \frac{f_n}{f_t} = -\frac{v^2}{\rho \dot{v}}.$$

(*e*) When the path is straight, that is, when $\rho = \infty$, the normal acceleration is nil; therefore in this case the total acceleration is identical with the tangential acceleration.

(*f*) When the path is circular and the speed constant, then $\rho = r$, the radius of the circle, and $\dot{v} = 0$; therefore

$$f = f_n = -\frac{v^2}{r}.$$

90. Radial and Transverse Components of Acceleration. — Let P (Fig. 58) be any point of the path at which the acceleration of the particle is to be considered. Take two neighboring points P_1 and P_2, and let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the velocities at these points. Then the change in the radial velocity in going from P_1 to P_2 is obtained by subtracting the radial component of $\mathbf{v}_1$ from that of $\mathbf{v}_2$. Replace $\mathbf{v}_1$ and $\mathbf{v}_2$ by their components along and at right angles to r_1 and r_2, respectively, and denote these components by $\mathbf{v}_{r_1}$, $\mathbf{v}_{p_1}$ and $\mathbf{v}_{r_2}$, $\mathbf{v}_{p_2}$; then it will be seen from the figure that

$$(v_{r_2} \cos \epsilon_2 - v_{p_2} \sin \epsilon_2) - (v_{r_1} \cos \epsilon_1 + v_{p_1} \sin \epsilon_1)$$

is the total change in the radial velocity. Therefore the radial component of the acceleration is

$$f_r = \operatorname*{limit}_{t=0} \left[\frac{v_{r_2} \cos \epsilon_2 - v_{p_2} \sin \epsilon_2 - v_{r_1} \cos \epsilon_1 - v_{p_1} \sin \epsilon_1}{t} \right]$$

where t is the time taken by the particle to go from P_1 to P_2. But as the points P_1 and P_2 approach P as a limit, the following substitutions become permissible.

$$\cos \epsilon_1 = \cos \epsilon_2 = 1^*, \quad \sin \epsilon_1 = \epsilon_1, \quad \sin \epsilon_2 = \epsilon_2.$$
$$v_{r_2} - v_{r_1} = dv_r, \quad v_{p_1} = v_{p_2} = v_p, \quad \epsilon_1 + \epsilon_2 = d\theta.$$

Making these substitutions in the expression for f_r, we obtain

$$\begin{aligned} f_r &= \underset{t \doteq 0}{\text{limit}} \left[\frac{(v_{r_2} - v_{r_1}) - v_p(\epsilon_1 + \epsilon_2)}{t} \right] \\ &= \frac{dv_r}{dt} - v_p \frac{d\theta}{dt} \\ &= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2, \end{aligned} \tag{XIV}$$

where θ is the angle r makes with the x-axis.

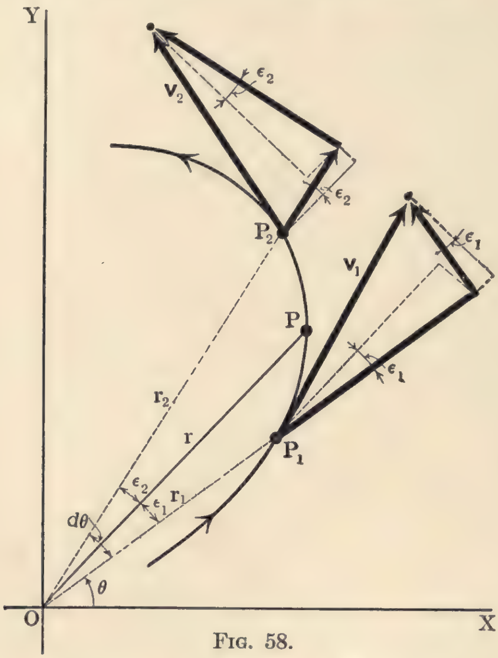


FIG. 58.

By similar reasoning we obtain the following expressions for the transverse acceleration, that is, the component of the acceleration along a perpendicular to the radius vector.

* See Appendix Avi.

$$f_p = \underset{t \doteq 0}{\text{limit}} \left[\frac{v_{r_2} \sin \epsilon_2 + v_{p_2} \cos \epsilon_2 - (-v_{r_1} \sin \epsilon_1 + v_{p_1} \cos \epsilon_1)}{t} \right]$$

$$= \underset{t \doteq 0}{\text{limit}} \left[\frac{v_r(\epsilon_1 + \epsilon_2) + (v_{p_2} - v_{p_1})}{t} \right]$$

$$= v_r \frac{d\theta}{dt} + \frac{dv_p}{dt}$$

$$= \omega \frac{dr}{dt} + \frac{d}{dt}(r\omega)$$

$$= \frac{1}{r} \frac{d}{dt}(r^2\omega), \qquad \text{(XV)}$$

where ω is the angular velocity of the radius vector.

PROBLEMS.

1. A particle describes the parabola $y = 2\,px$ so that its velocity along the x-axis is constant and equals u. Find the total velocity and the acceleration.

2. Discuss the motions defined by the following equations deriving the expressions for the path, velocity, acceleration, and the various components of the last two:

(a) $x = at, \quad y = b\sqrt{t}.$

(b) $x = at, \quad y = bt - \frac{1}{2}gt^2.$

(c) $x = ae^{kt}, \quad y = be^{kt}.$

(d) $x = at, \quad y = be^{-kt}.$

(e) $x = at, \quad y = b \sin \omega t.$

(f) $x = a \cos \omega t, \quad y = bt.$

3. Express in terms of t the velocity and the acceleration of a particle which moves so that $\dot{x} = ay$ and $\dot{y} = ax$.

4. A particle describes a circle of radius a with a constant speed v. Find $f_x, f_y, f_s, f_p, f_r, f_n, f_\tau$, and f.

5. Work out the preceding problem graphically.

91. Angular Acceleration.—Angular acceleration is the time rate at which angular velocity changes. Therefore, denoting it by γ, we have

$$\left.\begin{aligned} \gamma &= \frac{d\omega}{dt} = \dot{\omega} \\ &= \frac{d^2\theta}{dt^2} = \ddot{\theta}. \end{aligned}\right\} \qquad \text{(XVI)}$$

If the angular velocity of a body increases uniformly $\frac{1 \text{ rad.}}{1 \text{ sec.}}$ in one second the body is said to have a unit angular

acceleration. Therefore the unit is the $\frac{\text{rad.}}{\text{sec.}^2}$. The dimensions of angular acceleration are given by $[T^{-2}]$.

ILLUSTRATIVE EXAMPLE.

A particle moves so that the coördinates of its position at any instant are given by the equations

$$x = a \cos kt,$$
$$y = a \sin kt.$$

Find the acceleration and its components.

In a previous illustrative example, p. 82, it was shown that these equations represent uniform circular motion, with the following data:

$$v = ka, \qquad \omega = k,$$
$$v_x = -ka \sin kt, \qquad v_r = 0,$$
$$v_y = ka \cos kt, \qquad v_p = ka.$$

Therefore

$$f_x = -k^2 a \cos kt = -k^2 x, \qquad f_r = \frac{dv_r}{dt} - r\omega^2 = -k^2 a, \qquad f_r = \dot{v} = 0,$$
$$f_y = -k^2 a \sin kt = -k^2 y, \qquad f_p = \frac{1}{r}\frac{d}{dt}(r^2\omega) = 0, \qquad f_n = -\frac{v^2}{\rho} = -\frac{k^2a^2}{a}$$
$$f = -k^2 a, \qquad \omega = \frac{d}{dt}k = 0, \qquad = -ka^2.$$

It will be observed that f_r has a value different from zero, while v_r is nil.

PROBLEMS.

1. A flywheel making 250 revolutions per minute is brought to rest in 2 minutes. Find the average angular acceleration.

2. A flywheel making 250 revolutions per minute is retarded by a constant acceleration of $-5\,\frac{\text{rad.}}{\text{sec.}^2}$. How many revolutions will the flywheel make before stopping?

3. In the preceding problem find the time it takes the flywheel to come to rest.

4. Find the angular velocity and angular acceleration of a particle which moves in a manner defined by the following pairs of equations:

(a) $\rho = a \sin \omega t, \quad \theta = b \sin \omega t.$

(b) $\rho = a \sin \omega t, \quad \theta = b \cos \omega t.$

(c) $\rho = a \sin \omega t, \quad \theta = bt.$

(d) $\rho = ae^{bt}, \quad \theta = bt.$

5. In problem 4 find the equation of the path and plot it.

GENERAL PROBLEMS.

Find the velocity, the acceleration, and the path of a particle whose motion is defined by the following pair of equations:

1. $x = ae^{kt}, \qquad y = be^{-kt}.$
2. $x = a \sin \omega t, \qquad y = a \sin 2\omega t.$
3. $x = a \sin \omega t, \qquad y = b \cos 2\omega t.$
4. $x = a \sin(\omega t + \delta), \qquad y = b \sin(\omega t + \delta).$
5. $x = a \sin(\omega t + \delta), \qquad y = b \cos(\omega t + \delta).$
6. $x^2 = 4y, \qquad y = at^2.$
7. $x = 4at, \qquad y = bx.$
8. $x^2 = 4ay, \qquad y = a \sin \omega t.$
9. $x = a \cos \omega t, \qquad y^2 = 4ax.$
10. $x^2 + y^2 = a^2, \qquad \dot{x}^2 + \dot{y}^2 = b^2.$

CHAPTER VI.

MOTION OF A PARTICLE.

KINETIC REACTION.

92. Kinetic Reaction.—The Law of Action and Reaction, which we found so useful in studying equilibrium, is applicable not only to problems of equilibrium but also to those of motion. In applying it to motion, however, we must extend the meaning of the term "reaction" so as to include a form of reaction which is known as *kinetic reaction.* In order to understand the nature of kinetic reaction consider the following ideal experiment:

Suppose you hold one end of a long elastic string, the other end of which is attached to a rectangular block placed upon a perfectly horizontal and smooth table. Let another person pull the block along the plane of the table and thereby stretch the string. While the string is being stretched you have to exert a force on it in order to keep your end of it fixed. At any instant the force with which you pull the string equals and is opposite to the force with which the string pulls your hand. The action equals the reaction and is oppositely directed. The same is true about the block and the person who holds it. What will happen if the block is released? Will the force which the string exerts on your hand cease as soon as the block is released? No. The string pulls on until it regains its natural length, something which does not take place instantaneously. The elasticity of the string urges it to assume its natural length. But this cannot be accomplished without moving the block. Therefore the string moves the block. But in order to start the block the string must exert a force on it, and this in spite of the fact that the weight of

the block is exactly balanced by the reaction of the plane so that there are no forces to be overcome in order to move the block. Therefore we conclude that the block resists any attempt to start it into motion. In other words, the block offers resistance to a force which accelerates it. This resistance is the *kinetic reaction*.

In one respect kinetic reaction is similar to frictional and resisting forces, namely, it is not aggressive. The kinetic reaction of a body manifests itself only when its state of rest or of motion is interfered with. A body which is at rest or moving with a constant velocity does not display any kinetic reaction, but as soon as it is set in motion or its velocity is changed kinetic reaction appears; further the kinetic reaction of a body is greater the greater the acceleration imparted to it.

93. Generalization of the Law of Action and Reaction. — When the terms "action" and "reaction" are used so as to mean kinetic reactions as well as forces and torques, then the law is directly applicable to problems of motion as well as to problems of equilibrium. It will be remembered that in Chapter III the law was split into two sections, of which the second section is not applicable to single particles. Therefore we need to consider here only the first section, which states:

To every linear action there is always an equal and opposite linear reaction, or the sum of all the linear actions to which a body or a part of a body is subject at any instant vanishes.

$$\Sigma \mathbf{A}_l = 0. \qquad (\mathrm{A}_l)$$

If we replace the term "linear action" by the terms "force" and "linear* kinetic reaction" the law may be put into the following form.

* The adjective "linear" is introduced in order to distinguish between the kinetic reaction which is related to forces and the kinetic reaction introduced at the beginning of Chapter IX, which is related to torques.

The sum of all the forces acting upon a body plus the linear kinetic reaction equals zero, or the resultant of all the forces acting upon a body equals and is opposite to the linear kinetic reaction.

$$\left.\begin{array}{l}\textbf{Sum of all forces + linear kinetic reaction} = 0\\ \textbf{or Resultant force} = -(\textbf{linear kinetic reaction}).\end{array}\right\} \qquad (A_l')$$

Now let us apply the law to the experiment of the preceding section. After the block is released it is acted upon by three forces, namely, its weight, the reaction of the table, and the pull of the string. The law states that the resultant of these forces equals the kinetic reaction of the block and is oppositely directed. Since the weight and the reaction of the table are exactly balanced the pull of the string is the resultant force. Therefore the kinetic reaction of the block equals the pull of the string and has a direction opposite to that in which the block is pulled.

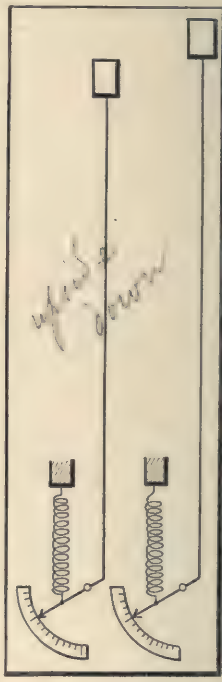


FIG. 59.

94. Definition of Mass. — In the block experiment suppose the free end of the string to be connected to a spring balance which is fixed on the table, Fig. 59. Further suppose the block to be set in motion as in the previous experiment. Let one person observe the readings of the balance and another the acceleration of the block. If F_1, F_2, F_3, etc., denote the readings of the balance and f_1, f_2, f_3, etc., the values of the acceleration of the block, obtained simultaneously with the readings of the balance, then it will be found that the following relations hold true:

$$\frac{F_1}{f_1} = \frac{F_2}{f_2} = \frac{F_3}{f_3} = \cdot \cdot \cdot = m, \qquad \text{(I)}$$

where m is a constant. But since the readings of the balance give the values of the kinetic reaction, equation (I) states that the kinetic reaction of the block is proportional to the acceleration. The constant of proportionality, m, is called the *mass* of the block. We have, therefore, the following definition for mass.

The mass of a body is a constant scalar magnitude which equals the quotient of the magnitude of the kinetic reaction of the body by the magnitude of its acceleration.

95. Measure of Kinetic Reaction. — Suppose we have several sets of apparatus consisting of a spring balance, a long elastic string, and a block, set up on a smooth horizontal table. Let two persons attend to each set of apparatus: one to observe the readings of the balance and the other to observe the acceleration of the block. Suppose the blocks to be set in motion as in the last experiment, and the pull registered by each balance observed at an instant when the corresponding block attains a certain definite acceleration f. Then if F_1, F_2, F_3, etc., denote the readings of the balances and m_1, m_2, m_3, etc., the masses of the blocks, it will be found that the following relations hold good:

$$\frac{F_1}{m_1} = \frac{F_2}{m_2} = \frac{F_3}{m_3} = \cdot \cdot \cdot = f. \qquad \text{(II)}$$

Equations (II) state that when bodies have equal accelerations their kinetic reactions are proportional to their masses.

Therefore equations (I) and (II) state that the kinetic reaction of a body is proportional to the product of its mass by its acceleration; that is,

$$\textit{kinetic reaction} = kmf, \qquad \text{(III)}$$

where k is the constant of proportionality. When the quantities involved in the last equation are measured in the same system of units the constant k becomes unity, in which case we have

$$\textit{kinetic reaction} = mf. \qquad \text{(III')}$$

If we want to express the fact that kinetic reaction and acceleration are oppositely directed we put the last equation in the vector notation and write,

$$\textbf{kinetic reaction} = -m\mathbf{f}. \qquad \text{(IV)}$$

Equations (I) and (II) and consequently equation (IV) hold good not only when the acceleration is due to a change in the magnitude of the velocity but also when it is due to a change in its direction. As an illustration of this fact consider the following ideal experiment:

Let P, Fig. 60, be a particle attached to the end of an inextensible string, which passes through the hole O, in the middle of the smooth and horizontal table A, and is fastened to the spring balance S. If we project the particle in the plane of the table in a direction at right angles to the line OP we will find that it describes a circle about the point O, with a speed equal to the speed of projection. We will further observe that the balance registers a pull.

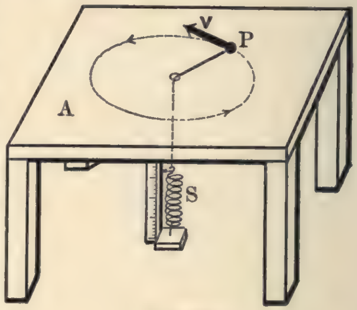


Fig. 60.

Now let us examine the forces experienced by the particle during its motion. The particle is acted upon by three forces, namely, its weight, the reaction of the table, and the pull of the string. Since the surface of the table is perfectly smooth and horizontal the weight and the reaction of the plane exactly balance each other. Therefore the pull of the string is the resultant force. Thus the particle is pulled toward the point O, but somehow manages to keep the same distance from it; and this in spite of the fact that it is not acted upon by forces which would counterbalance the pull of the string. The explanation is plain. While describing the circle the direction of the velocity of the

particle is continually changing, that is, the particle is being accelerated. Therefore kinetic reaction manifests itself and acts in a direction opposed to that of the acceleration, that is, away from the point O. Hence the pull of the string. But the pull which comes into play is just enough to overcome the kinetic reaction, therefore the particle neither approaches to, nor recedes from, the point O.

Suppose we project the particle with different velocities, observe the corresponding readings of the spring balance, and compute the accelerations from $-\frac{v^2}{\rho}$, the expression for the normal acceleration, p. 94. Let F_1, F_2, F_3, etc., denote the readings of the balance and f_1, f_2, f_3, etc., denote the accelerations; then we shall find that the relations between the accelerations and the readings of the balance are given by equations (I).

On the other hand if we fasten particles of different masses to the string and give them equal accelerations, we shall find that equations (II) hold true. Therefore we conclude that whether the acceleration be due to changes in the magnitude of the velocity, or in the direction, or in both, the kinetic reaction equals the product of the mass by the acceleration and is opposed to the latter.

The kinetic reaction of the last experiment may be differentiated from that of the experiments of sections 92 and 94 by emphasizing the fact that the former comes into play when there is a normal acceleration, while the latter manifests itself whenever there is acceleration along the tangent. The resultant or total kinetic reaction is the vector sum of the two.

The results of the last few sections may be summed up in the following manner:

(a) *The tangential kinetic reaction has a magnitude $m\dot{v}$ and has a direction opposite to that of the tangential acceleration,*

$$\textbf{Tangential kinetic reaction} = -m\dot{\mathbf{v}}_\tau. \qquad \text{(IV')}$$

(b) *The normal kinetic reaction has a magnitude* $\frac{mv^2}{\rho}$ *and has a direction opposite to that of the normal acceleration. In other words it has the same direction as the radius of curvature, i.e., away from the center of curvature,*

$$\textbf{Normal kinetic reaction} = \frac{mv^2}{\rho}. \qquad \text{(IV}''\text{)}$$

(c) *The total kinetic reaction has a magnitude* $m\sqrt{\dot{v}^2 + \frac{v^4}{\rho^2}}$ *and has a direction opposite to that of the total acceleration,*

$$\textbf{Total kinetic reaction} = -m\dot{\mathbf{v}}. \qquad \text{(IV)}$$

FORCE EQUATION.

96. Force Equation. — Combining (A_l') and (IV) and denoting the resultant force by $\mathbf{F}$ we obtain

$$\left.\begin{aligned} \mathbf{F} &= m\mathbf{f}, \\ &= m\dot{\mathbf{v}}. \end{aligned}\right\} \qquad \text{(V)}$$

Equation (V) is called the *force equation.* It states that the resultant force acting upon a particle equals the product of the mass by the acceleration and has the same direction as the latter.

Since the magnitude of $\dot{\mathbf{v}}$ is $\sqrt{\dot{v}^2 + \frac{v^4}{\rho^2}}$, the force equation takes the following form when stripped of its vector notation:

$$F = m\sqrt{\dot{v}^2 + \frac{v^4}{\rho^2}}. \qquad \text{(VI)}$$

In equation (VI) $\dot{v}$ represents that part of the acceleration which is due to the change in the magnitude of the velocity and $\frac{v^2}{\rho}$ represents that part which is due to the change in the direction.*

97. Component-force Equations. — Splitting equation (V) into two component equations which correspond to the di-

* See p. 94 for the tangential and normal components of $\dot{\mathbf{v}}$.

rections of the tangent and the normal and then dropping the vector notation we obtain

$$F_\tau = m\dot{v}, \tag{VII}$$

and

$$F_n = -m\frac{v^2}{\rho}. \tag{VIII}$$

The negative sign in equation (VIII) states that the normal component of the resultant force, and consequently the resultant force itself, is directed toward the concave part of the path. The last two equations may be obtained directly from (VI) by considering them as the force equations for special cases of motion. Thus when the path of the moving particle is a straight line $\rho = \infty$, and consequently

$$F = m\dot{v}. \tag{VII$'$}$$

On the other hand when the particle moves with a constant speed $\dot{v} = 0$, and therefore

$$F = -m\frac{v^2}{\rho}. \tag{VIII$'$}$$

If in addition the radius of curvature of the path does not change, that is, if the particle moves in a circle with a constant speed, then

$$F = -m\frac{v^2}{r}, \tag{VIII$''$}$$

where r is the radius of the circle.

The following is a useful set of component-force equations obtained by splitting equation (V) into three component equations which correspond to the directions of the axes of a rectangular system:

$$\left.\begin{aligned} F_x &= m\ddot{x}, \\ F_y &= m\ddot{y}, \\ F_z &= m\ddot{z}. \end{aligned}\right\} \tag{IX}$$

Equations (IX) emphasize the fact that *the component of the resultant force along any direction equals the product of*

the mass by the component of the acceleration along the same direction.

98. Equilibrium as a Special Case of Motion. — When the right-hand member of the force equation vanishes, that is, when the acceleration is nil, the resultant force vanishes. But this is the condition of the equilibrium of a particle, therefore equilibrium is a case of motion in which acceleration is zero. For the equilibrium of a particle it is necessary that the resultant force vanish, but this condition is not sufficient because while the acceleration vanishes when $F = 0$, the velocity may have any constant value. In other words a particle may be in motion even when the resultant of the forces which act upon it vanishes. Therefore in order that a particle stay at rest not only must the resultant of the forces vanish but it must be at rest at the time of application of these forces.

99. Dimensions of Force. — In discussing the equilibrium of bodies we only compared forces because it was all that was necessary; besides we had no means of expressing forces in terms of other physical magnitudes. But now the force equation enables us to express forces in terms of the three fundamental magnitudes and thus to connect them with other physical quantities.

If we substitute the dimensions of mass and acceleration in the force equation we obtain the following dimensional formula for force:

$$[F] = [MLT^{-2}].$$

100. Units of Force. — The C.G.S. unit of force is the *dyne*. It is a force which gives a body of one gram mass a unit acceleration. This is denoted symbolically by the following formula:

$$\text{dyne} = \frac{\text{gm. cm.}}{\text{sec.}^2}.$$

The British unit of force is the pound, which we have already defined (p. 76) as the weight, in London, of a body

which has a mass of about 453.6 gms. The weight of a body is the force with which it is attracted towards the center of the earth. Therefore if m denotes the mass of a body and g the magnitude of the acceleration which the gravitational attraction of the earth imparts to bodies, the force equation gives us

$$W = mg, \tag{X}$$

where W is the weight of the body. The value of g is slightly different at different points of the surface of the earth. It is greatest at the poles and least at the equator. The maximum variation, however, is less than one per cent; therefore for most purposes it may be considered as constant. For engineering problems $32.2\ \frac{\text{ft.}}{\text{sec.}^2}$ or $981\ \frac{\text{cm.}}{\text{sec.}^2}$ are close enough approximations to the actual value of g in any locality.

The relation between the pound and the dyne may be obtained by the help of equation (X). Thus

$$\begin{aligned} 1\text{ lb.} &= 1\text{ pd.} \times 32.2\frac{\text{ft.}}{\text{sec.}^2} \\ &= 32.2\frac{\text{pd. ft.}}{\text{sec.}^2} \\ &= 4.45 \times 10^6\frac{\text{gm. cm.}}{\text{sec.}^2} \\ &= 4.45 \times 10^6 \text{ dynes,} \end{aligned}$$

where "lb." is the symbol for the pound (weight) and "pd." the symbol for the mass of a body which weighs one pound. In order to emphasize the distinction between the two they are often called pound-weight and pound-mass.

101. Difference between Mass and Weight. — The beginner often finds it difficult to distinguish between the mass of a body and its weight. He is apt to ask such a question as this, "When I buy a pound of fruit what do I get, one

pound-mass or one pound-weight?"* The difficulty is due to the fact that the common methods for comparing the masses of bodies make use of their weights.

There are two general methods by which masses may be compared, both of which are based upon the force equation. Let F_1 and F_2 be the resultant forces acting upon two bodies having masses m_1 and m_2, and f_1 and f_2 be the accelerations produced. Then the force equation gives

$$F_1 = m_1 f_1,$$

$$F_2 = m_2 f_2,$$

and

$$\frac{m_1}{m_2} = \frac{F_1}{F_2} \cdot \frac{f_2}{f_1}.$$

(1) If the forces are of such magnitudes that the accelerations are equal then the masses are proportional to the forces; for when $f_1 = f_2$, the last equation becomes

$$\frac{m_1}{m_2} = \frac{F_1}{F_2}.$$

This gives us a method of comparing masses, of which the common method of weighing is the most important example.

* This question may be answered in the following manner. "The fruit which you get has a mass of 1 pd. (about 453.6 gm.) and which weighs 1 lb. (about 4.45×10^6 dynes). If the fruit could be shipped to the moon during the passage the weight would diminish down to nothing and then increase to about one-sixth of a pound. The zero weight would be reached at a point about nine-tenths of the way over. Up to that position the weight would be with respect to the earth, that is, the fruit would be attracted towards the earth; but from there on the weight would be with respect to the moon. The mass of the fruit, however, would be the same on the earth, during the passage, and on the moon. It would be the same with respect to the moon as it is with respect to the earth. Mass is an intrinsic property of matter, therefore it does not change. Weight is the result of gravitational attraction; consequently it depends upon, (a) the body which is attracted, (b) the bodies which attract it, and (c) the position of the former relative to the latter. It is evident therefore that when a body is moved relative to the earth its weight changes."

If W_1 and W_2 denote the weights of two bodies of masses m_1 and m_2, then by equation (X) we obtain

$$W_1 = m_1 g,$$
$$W_2 = m_2 g,$$

and
$$\frac{m_1}{m_2} = \frac{W_1}{W_2},$$

where g is the common acceleration due to gravitational attraction.

(2) If the forces acting upon the bodies are equal the masses are inversely proportional to the accelerations:

$$\frac{m_1}{m_2} = \frac{f_2}{f_1}.$$

This gives us the second method by which masses may be compared. The following are more or less practicable applications of this method:

(a) Let A and B (Fig. 61) be two bodies connected with a long elastic string of negligible mass, placed on a perfectly

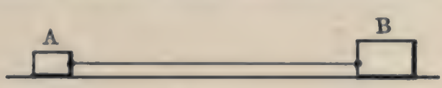


Fig. 61.

smooth and horizontal table. Suppose the string to be stretched by pulling A and B away from each other. It is evident that when the bodies are released they will be accelerated with respect to the table and that the accelerating force, that is, the pull of the string, will be the same for both bodies. Therefore if f_1 and f_2 denote their accelerations at any instant of their motion, the ratio of their masses is given by the relation

$$\frac{n_1}{m_2} = \frac{f_2}{f_1}.$$

(b) Suppose the bodies whose masses are to be compared

to be fitted on a smooth horizontal rod (Fig. 62) so that they are free to slide along it. If the rod is rotated about a vertical axis the bodies fly away from the axis of rotation. If, however, the bodies are connected by a string of negligible mass they occupy positions on the two sides of the axis, which depend upon the ratio of the masses. So far as the motion along the rod is concerned, each body is equivalent to a particle of the same mass placed at the center of mass of the body.*

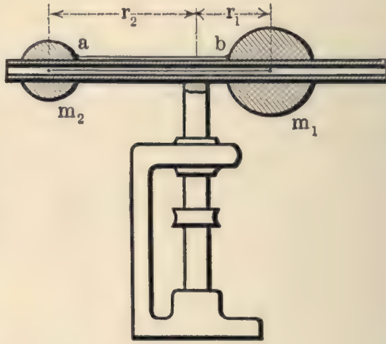


Fig. 62.

Suppose, as it is assumed in Fig. 62, the horizontal rod to be hollow and to have smooth inner wall; further suppose the centers of mass of the given bodies to lie on the axis of the rod. Then if at the center of mass of each body a particle of equal mass is placed and the two particles connected by means of a massless string of proper lengths, the positions of the particles will remain at the centers of mass of the given bodies even when the rod is set rotating about the vertical axis.

Now let m_1 and m_2 be the masses of the particles and f_1 and f_2 their accelerations due to the rotation of the tube about the vertical axis. Then since the tensile force in the string is the same at its two ends, the forces acting upon the particles are equal. Therefore we have

$$F = m_1 f_1 = m_2 f_2,$$

or

$$\frac{m_1}{m_2} = \frac{f_2}{f_1}.$$

* For a proof of this statement see p. 242.

But if r_1 and r_2 denote the distances of the particles from the axis of rotation, and P the period of revolution, then

$$f_1 = -\frac{v_1^2}{r_1} = -\frac{4\pi^2 r_1}{P^2} \quad \text{and} \quad f_2 = -\frac{v_2^2}{r_2} = -\frac{4\pi^2 r_2}{P^2}.$$

Therefore
$$\frac{m_1}{m_2} = \frac{r_2}{r_1}$$

gives the ratio of the masses of the particles as well as those of the given bodies.

MOTION OF A PARTICLE UNDER A CONSTANT FORCE.

102. Case I. Rectilinear Motion. — Suppose a particle of mass m to be acted upon by a force **F**, which is constant in direction as well as in magnitude. Then the force equation gives

$$m\frac{dv}{dt} = F, \tag{I}$$

or
$$\frac{dv}{dt} = \frac{F}{m} = f. \tag{I$'$}$$

Since both m and F are constant, f, the acceleration, is also constant. Integrating equation (I$'$) once we obtain

$$v = ft + c,$$

where c is a constant to be determined by the initial conditions of the motion. Let the initial velocity be denoted by v_0; then $v = v_0$, when $t = 0$, therefore $c = v_0$ and

$$v = v_0 + ft. \tag{1}$$

Substituting $\frac{ds}{dt}$ for v in equation (1) and integrating,

$$s = v_0 t + \tfrac{1}{2} ft^2 + c'.$$

Let $s = 0$, when $t = 0$; then $c' = 0$. Therefore

$$s = v_0 t + \tfrac{1}{2} ft^2. \tag{2}$$

Eliminating t between equations (1) and (2) we get

$$v^2 = v_0^2 + 2fs. \tag{3}$$

103. Equations of Motion. — The force equation and the equations (1), (2), and (3), which connect v, s, and t are called *equations of motion.* The force equation will be called the *differential equation of motion,* while those which are obtained by integrating the force equation will be called the *integral equations of motion.*

104. Special Cases: A. Motion when the Force is Zero.— When the force vanishes the acceleration is zero. Therefore equations (1) to (3) become

$$v = v_0 = \text{const.},$$
$$s = v_0 t.$$

Therefore the particle moves in a straight path with unchanging velocity.

105. B. Falling Bodies.—The force experienced by a falling body is its weight $m\mathbf{g}$. Therefore the acceleration of the motion is $\mathbf{g}$, the gravitational acceleration due to the attraction of the earth. So long as the distance through which the body falls is very small compared with the radius of the earth, $\mathbf{g}$ may be considered to remain constant. Therefore the motion of falling bodies may be treated as a special case of rectilinear motion under a constant force. Hence the equations of motion of a falling body are obtained by replacing f by g in equations (1) to (3). Making this substitution we get

$$v = v_0 + gt,$$
$$s = v_0 t + \tfrac{1}{2} gt^2,$$
$$v^2 = v_0{}^2 + 2\,gs.$$

When a body falls from rest the initial velocity is zero. Therefore we must put $v_0 = 0$ in the last three equations before using them for bodies falling from rest.

When a body is projected vertically upward the acceleration is in the opposite direction from the velocity; in other words, it is negative. Therefore in the last three equations

g must be replaced by $(-g)$ before they are applied to the motion of bodies which are projected vertically upwards.

PROBLEMS.

1. A steel plate weighing 10 pounds is placed on a perfectly smooth and perfectly horizontal sheet of ice. The plate is then moved by means of a string, one end of which is fastened to the plate, and the string is pulled in a direction parallel to the surface of the ice. Is it necessary to apply a force to the string in order to give the plate a desired velocity? Why? What will the magnitude of the force depend upon?

2. In the preceding problem the block is given a velocity of $100 \frac{\text{ft.}}{\text{sec.}}$ in 5 seconds. Find the tension of the string supposing it to be constant. How far will the plate have traveled in the meantime?

3. In the preceding problem the string is just strong enough to support half the weight of the plate. What is the shortest time in which the plate can be pulled through a distance of 162 feet?

4. In the preceding problem suppose the contact to be rough and to have a coefficient of friction equal to 0.1.

5. A bullet is fired with a muzzle velocity of $500 \frac{\text{meters}}{\text{sec.}}$. Find the average acceleration, supposing the length of the barrel to be 80 cm.

6. A stone is sent gliding over a horizontal sheet of ice with a speed of $10 \frac{\text{meters}}{\text{sec.}}$. How far and how long will it move if the coefficient of friction is 0.1?

7. An elevator starts from rest and rises to a height of 100 feet in 10 seconds, with a constantly increasing velocity. Find the increase in pressure exerted on the feet of a man in the elevator who weighs 150 pounds.

8. A man can just lift 350 pounds when on the ground. How much can he lift when in an elevator descending with an acceleration of $4 \frac{\text{ft.}}{\text{sec.}^2}$?

9. An elevator starts from rest and rises 100 feet in five seconds, with a constant acceleration. Find the tension of the rope which pulls it up if the elevator weighs 2000 pounds; neglect the frictional forces.

10. A body is projected vertically upward with a velocity of $50 \frac{\text{ft.}}{\text{sec.}}$ at the edge of a pit 200 feet deep. When will it strike the bottom?

11. What is the lowest level, over the enemy's camp, to which a balloon can safely descend, if the enemy is provided with guns which have muzzle velocities of 2000 feet per second?

12. A body, which is dropped from the top of a tower, strikes the ground half a second after it passes by a window 84 feet above the sidewalk. Find the height of the tower.

13. In the preceding problem find the velocity with which the body strikes the ground.

14. A man weighing 150 pounds is obliged to leave his room by way of a window 50 feet above the sidewalk. He has a rope which is long enough but cannot support more than 125 pounds. What is the least velocity with which he can reach the ground?

15. A ball is dropped in an elevator from a point 6 feet above the floor of the elevator. How long will it take to strike the floor if the elevator is descending with a constant speed of $10 \frac{\text{ft.}}{\text{sec.}}$?

106. C. Motion of a Particle along a Smooth Inclined Plane. — There are two forces acting on the particle, its weight and the reaction of the plane. The weight is $m\mathbf{g}$ and acts downwards. The reaction of the plane, **N**, is normal to the plane, because the plane is smooth. Therefore setting the components of the kinetic reaction along and at right angles to the plane equal to the sum of the corresponding components of the forces we obtain

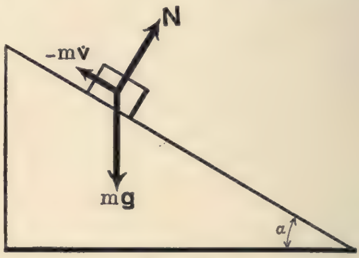


FIG. 63.

$$m\frac{dv}{dt} = mg \sin \alpha,$$

and
$$0 = N - mg \cos \alpha.$$

The last equation states that forces along the normal add up to zero and therefore do not affect the motion. We have, therefore, to consider only the first equation, which gives

$$\frac{dv}{dt} = g \sin \alpha = \text{const.}$$

Therefore the equations of motion are obtained by substituting $g \sin \alpha$ for f in equations (1) to (3). Thus we have

$$v = v_0 + gt \sin \alpha,$$
$$s = v_0 t + \tfrac{1}{2} gt^2 \sin \alpha,$$
$$v^2 = v_0^2 + 2\, gs \sin \alpha,$$

for the equations of motion.

PROBLEMS.

1. A number of particles slide down smooth inclined planes of equal height. Show that the time taken by each particle to reach the base is proportional to the length of the plane along which it slides.

2. Given the base of an inclined plane, find the height so that the horizontal component of the velocity acquired in descending it may be greatest possible.

3. Two particles are projected simultaneously, one up and the other down a smooth inclined plane. Find the velocities of projection if the particles pass each other at the middle of the plane.

4. Show that the time taken by a particle to slide down any chord which begins at the highest point of a vertical circle is constant and equals $2\sqrt{\frac{a}{g}}$, where a is the radius of the circle.

5. A particle is projected down an inclined plane of length l and height h. At the same time another particle is let fall vertically from the same point. Find the velocity of the projection of the first particle if both strike the base at the same time.

6. A ship stands at a distance d from its pier. Show that the length of the chute which will make the time of sliding down it a minimum is $d\sqrt{2}$.

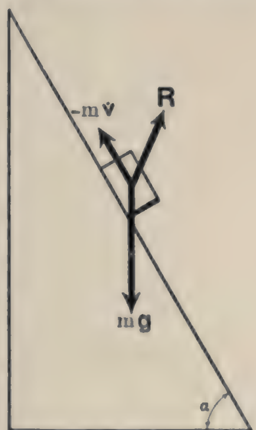


FIG. 64.

107. D. Motion of a Particle along a Rough Inclined Plane. — The only difference between this problem and the last one is that the reaction of the plane is not normal to the surface. On account of friction the reaction **R** has a component along the plane. Denoting

the frictional component of **R** by **F** and the normal component by **N**, and equating the components of the kinetic reaction along and at right angles to the plane to the sums of the corresponding components of the forces we obtain

$$m\frac{dv}{dt} = mg \sin \alpha - F, \tag{a}$$

$$0 = N - mg \cos \alpha. \tag{b}$$

But if μ is the coefficient of friction then

$$\begin{aligned} F &= \mu N & &\text{[p. 22]} \\ &= \mu\, mg \cos \alpha & &\text{[by (b)].} \end{aligned}$$

Substituting this value of F in equation (a) we obtain

$$m\frac{dv}{dt} = mg \sin \alpha - \mu\, mg \cos \alpha,$$

or

$$\frac{dv}{dt} = g\,(\sin \alpha - \mu \cos \alpha).$$

Thus the acceleration is constant. Therefore the equations of motion are obtained by replacing f by $g\,(\sin\,\alpha - \mu \cos\,\alpha)$ in equations (1) to (3) of page 113:

$$\begin{aligned} v &= v_0 + gt\,(\sin \alpha - \mu \cos \alpha), \\ s &= v_0 t + \tfrac{1}{2}\, gt^2\,(\sin \alpha - \mu \cos \alpha), \\ v^2 &= v_0{}^2 + 2\, gs\,(\sin \alpha - \mu \cos \alpha). \end{aligned}$$

PROBLEMS.

1. A car weighing 10 tons becomes uncoupled from a train which is moving down a grade of 1 in 200 at the rate of 50 miles per hour. If the frictional resistance is 15 pounds per ton, find the distance the car will travel before coming to rest.

2. The pull of a locomotive is 2500 pounds. Find the velocity it can give in 5 minutes to a train which weighs 75 tons. Take 10 pounds per ton for the resistance and consider the tracks to be horizontal.

3. In the preceding problem suppose the tracks to have a grade of 1 in 200 and find the velocity (a) going down grade and (b) going up grade.

108. E. Atwood's Machine. — The problem is to find the equations of motion of two particles connected by means of a string of negligible mass which is slung over a smooth pulley.

Let m_1 and m_2 be the masses of the particles. Then considering each particle separately we obtain the following for the force equations:

$$m_1 \frac{dv}{dt} = -T + m_1 g, \qquad \text{(a)}$$

$$m_2 \frac{dv}{dt} = T - m_2 g, \qquad \text{(b)}$$

where T is the tensile force in the string. Eliminating T between equations (a) and (b) we obtain

$$(m_1 + m_2) \frac{dv}{dt} = (m_1 - m_2)\, g, \qquad \text{(c)}$$

$$\frac{dv}{dt} = \frac{m_1 - m_2}{m_1 + m_2} g. \qquad \text{(d)}$$

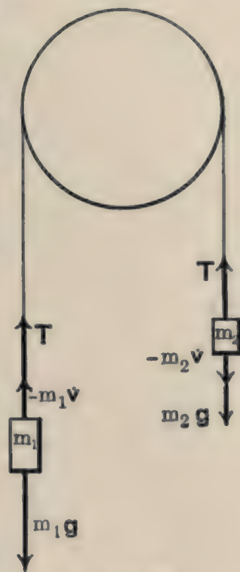


FIG. 65.

Therefore the acceleration is constant and consequently the equations of motion are obtained by substituting this value of the acceleration in equations (1) to (3) of page 113:

$$v = v_0 + \frac{m_1 - m_2}{m_1 + m_2} gt,$$

$$s = v_0 t + \frac{m_1 - m_2}{2(m_1 + m_2)} gt^2,$$

$$v^2 = v_0^2 + 2 \frac{m_1 - m_2}{m_1 + m_2} gs.$$

Eliminating $\frac{dv}{dt}$ between equations (a) and (b) we have

$$T = \frac{2\, m_1 m_2}{m_1 + m_2} g. \qquad \text{(e)}$$

Discussion. — Instead of considering the masses separately we can consider them as a single moving system and write a single force equation. Thus

(total moving mass) $\times$ (acceleration) = sum of the forces,

or
$$(m_1 + m_2)\frac{dv}{dt} = m_1g - m_2g,$$

which is identical with equation (c).

PROBLEMS.

1. Two particles of mass m_1 and m_2 are suspended by a string which is slung over a smooth table. A third particle of mass m_3 is attached to that portion of the string which is on the table. Prove that when the system is left to itself it will move with an acceleration of

$$\frac{m_1 - m_2}{m_1 + m_2 + m_3}g.$$

2. In the preceding problem suppose the table to be rough and find the acceleration. $\mu = 0.5$.

3. Discuss Atwood's machine supposing a frictional force to act between the string and the pulley (the latter is supposed to be fixed); take the frictional force to be equal to the tensile force in that portion of the string which is moving up.

109. Case II. Parabolic Motion, or Motion of Projectiles. — Consider the motion of a particle which is projected in a direction making an angle α with the horizon. When we neglect the resistance of the air, the only force which acts upon the particle is its weight, $m\mathbf{g}$. Taking the plane of motion to be the xy-plane, Fig. 66, we have

$$m\frac{d\dot{x}}{dt} = 0, \quad \text{or} \quad \frac{d\dot{x}}{dt} = 0, \tag{1}$$

$$m\frac{d\dot{y}}{dt} = -\,mg, \quad \text{or} \quad \frac{d\dot{y}}{dt} = -\,g, \tag{2}$$

where $\dfrac{d\dot{x}}{dt}$ and $\dfrac{d\dot{y}}{dt}$ are the components of the acceleration along the axes. Integrating equations (1) and (2) we get

$$\dot{x} = c_1,$$

and
$$\dot{y} = -\,gt + c_2.$$

Therefore the component of the velocity along the x-axis remains constant, while the component along the y-axis changes uniformly. Let v_0 be the velocity of projection, then when $t = 0$, $\dot{x} = v_0 \cos \alpha$ and $\dot{y} = v_0 \sin \alpha$. Making these substitutions in the last two equations we obtain

$$c_1 = v_0 \cos \alpha,$$

and
$$c_2 = v_0 \sin \alpha.$$

Therefore
$$\dot{x} = v_0 \cos \alpha, \tag{3}$$

and
$$\dot{y} = v_0 \sin \alpha - gt. \tag{4}$$

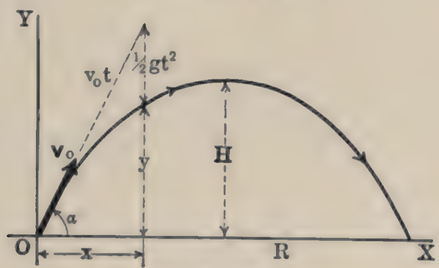


FIG. 66.

Therefore the total velocity at any instant is

$$\left.\begin{aligned} v &= \sqrt{\dot{x}^2 + \dot{y}^2} \\ &= \sqrt{v_0^2 - 2\, v_0 g \sin \alpha \cdot t + g^2 t^2} \end{aligned}\right\} \tag{5}$$

and makes an angle θ with the horizon defined by

$$\tan \theta = \frac{\dot{y}}{\dot{x}} = \frac{v_0 \cos \alpha}{v_0 \sin \alpha - gt}. \tag{6}$$

Integrating equations (3) and (4) we obtain

$$x = v_0 \cos \alpha \cdot t + c_3,$$

and
$$y = v_0 \sin \alpha \cdot t - \tfrac{1}{2} gt^2 + c_4.$$

But when $t = 0$, $x = y = 0$, therefore $c_3 = c_4 = 0$, and consequently

$$x = v_0 \cos \alpha \cdot t, \tag{7}$$

$$y = v_0 \sin \alpha \cdot t - \tfrac{1}{2} gt^2. \tag{8}$$

It is interesting to note that the motions in the two directions are independent. The gravitational acceleration does not affect the constant velocity along the x-axis, while the motion along the y-axis is the same as if the body were projected vertically with a velocity $v_0 \sin \alpha$. The projectile virtually rises a distance of $v_0 t \sin \alpha$ on account of its initial vertical velocity, and falls a distance $\frac{1}{2} gt^2$ on account of the gravitational acceleration.

THE PATH. — The equation of the path may be obtained by eliminating t between equations (7) and (8). This gives

$$y = x \tan \alpha - \frac{g}{2 v_0^2 \cos^2 \alpha} x^2, \tag{9}$$

which is the equation of a parabola.

THE TIME OF FLIGHT. — When the projectile strikes the ground its y-coördinate is zero. Therefore substituting zero for y in equation (8) we get for the time of flight

$$T = \frac{2 v_0 \sin \alpha}{g}. \tag{10}$$

THE RANGE. — The range, or the total horizontal distance covered by the projectile, is found by replacing t in equation (7) by the value of T, or by letting $y = 0$ in equation (9). By either method we obtain

$$\left.\begin{aligned} R &= \frac{2 v_0^2 \sin \alpha \cos \alpha}{g} \\ &= \frac{v_0^2 \sin 2 \alpha}{g}. \end{aligned}\right\} \tag{11}$$

Since v_0 and g are constants the value of R depends upon α. It is evident from equation (11) that R is maximum when $\sin 2 \alpha = 1$, or when $\alpha = \frac{\pi}{2}$. The maximum range is, therefore,

$$R_m = \frac{v_0^2}{g}. \tag{12}$$

the bodies are given the same angular velocity about a vertical axis through the point of suspension the particles will lie in the same horizontal plane.

6. If the masses in the preceding problem are equal how will the tensile force vary with the length of the strings?

7. Supposing the earth to be spherical discuss the variation in the weight of a body due to the rotation of the earth about its axis.

8. The moon describes a circular path around the earth once in every 27 days, 7 hours, and 43 minutes; find the acceleration at the center of the moon due to the attraction of the earth. Take 240,000 miles for the radius of the moon's orbit.

9. If the earth rotated fast enough to make the weights of bodies vanish at the equator show that the plumb line at any latitude would become parallel to the axis of the earth.

10. In the preceding problem what would the length of the day be?

11. How much would the weight of a body be increased at latitude 30° if the earth stopped rotating?

12. A particle suspended from a fixed point by a string of length l is projected horizontally with a speed $\sqrt{\frac{7}{2}lg}$; show that the string will become slack when the particle has risen to a height $\frac{3}{2}l$.

13. How much should the outer rail of a railroad track be raised at a curve in order that there be no lateral pressure on the rails when a train makes the curve at the rate of a mile a minute? The radius of the curve is 1500 feet and the distance between the tracks is 4 feet 8½ inches.

14. Prove that a locomotive will upset if it takes a curve with a speed greater than $\sqrt{\dfrac{gar}{2h}}$, on tracks the outer rails of which are not raised, where g denotes the gravitational acceleration, r the radius of the curve, a the distance between the rails, and h the height of the center of mass of the locomotive above the tracks.

15. Show that if there is no lateral pressure on the outer rails, while a car takes a curve, the relation

$$\tan\theta = \frac{v^2}{gr}$$

is satisfied, where θ is the angle the floor of the car makes with the horizon, v is the speed of the car, and r the radius of the curve.

111. II. Bodies Falling from Great Distances. — When the distance from which a body falls is not negligible compared with the radius of the earth the gravitational acceleration cannot be considered as constant during the fall. Therefore

the variation of the gravitational attraction must be taken into account. According to the law of gravitational attraction the force between two gravitating spherical bodies is of the following form:

$$F = -\gamma \frac{mm'}{r^2}, \tag{1}$$

where m and m' are the masses of the spheres, r is the distance between their centers, and γ is a positive constant. The negative sign indicates the fact that r is measured in a direction opposed to that in which F acts. When the gravitating bodies are the earth and a body which is small compared with the earth $\gamma = \frac{ga^2}{M}$, where M is the mass of the earth, a its radius, and g the gravitational acceleration on the surface of the earth. In order to show this observe that when the body is on the surface of the earth, that is, when $r = a$, the force is $-mg$, the weight of the body. Therefore replacing in equation (1) F by $-mg$ and m' by M and solving for γ we obtain

$$\gamma = \frac{ga^2}{M}. \tag{2}$$

Substituting in equation (1) this value of γ we get

$$F = -\frac{mga^2}{r^2} \tag{3}$$

for the force which acts upon a body of mass m during its fall towards the earth.

Therefore the force equation is

$$m\frac{dv}{dt} = -\frac{mga^2}{r^2}. \tag{4}$$

Dropping m from both sides of equation (4) and writing $v\frac{dv}{dr}$ for $\frac{dv}{dt}$ we obtain

$$v\frac{dv}{dr} = -\frac{ga^2}{r^2}. \tag{5}$$

Separating the variables and integrating we have

$$v^2 = \frac{2\,ga^2}{r} + c.$$

Now suppose the body starts to fall from a distance r' from the center, then $v = 0$ when $r = r'$ and $c = -\frac{ga^2}{r'}$. Therefore

$$v^2 = 2\,ga^2\left(\frac{1}{r} - \frac{1}{r'}\right) \tag{6}$$

gives the velocity at a distance r from the center. When the body falls from an infinite distance $r' = \infty$ and the velocity at any distance is

$$v_\infty = a\sqrt{\frac{2\,g}{r}}. \tag{7}$$

Therefore the velocity with which it will reach the surface of the earth is

$$\left.\begin{aligned} v_\infty &= \sqrt{2\,ga} \\ &\doteqdot 7\,\frac{\text{miles}}{\text{sec.}}. \end{aligned}\right\} \tag{8}$$

If the body starts to fall from a distance above the surface equal to the radius of the earth, then in equation (6) $r' = 2\,r$. Therefore

$$\begin{aligned} v &= \sqrt{ag}, \\ &\doteqdot 4.95\,\frac{\text{miles}}{\text{sec.}}. \end{aligned}$$

Therefore about seventy-one per cent of the velocity attained in falling from an infinite distance is developed in the last 4000 miles.

PROBLEMS.

1. A meteorite falls to the earth. Supposing it to start from infinity find the time it takes to travel the last 4000 miles.

2. A particle is attracted towards a fixed point by a force which varies inversely as the cube of the distance of the particle from the fixed point. Find the time it will take the particle to fall to the point if it starts from a distance d.

3. Discuss the motions of a particle which is repelled from a fixed point with a force which varies directly as the distance of the particle from the fixed point.

112. III. Motion of a Particle in a Resisting Medium. — As a concrete example of motion in a resisting medium consider the motion of falling bodies, taking the resistance of the atmosphere into account. At any instant of the motion two forces act on the body, i.e., the weight of the body and the resistance of the air. Denoting the resisting force by F we get

$$m\frac{dv}{dt} = mg - F \tag{1}$$

for the force equation. In order to be able to integrate the last equation we must make an assumption as to the nature of F.

Case I. Resistance Proportional to the Velocity. — Suppose F to be proportional to the velocity, then

$$F = k_1 v,$$

where k_1 is a positive constant. Substituting in the force equation this expression for F we obtain

$$m\frac{dv}{dt} = mg - k_1 v_1,$$

or

$$\frac{dv}{dt} = g - kv, \tag{2}$$

where $k = \frac{k_1}{m}$. Rearranging the last equation

$$\frac{dv}{v - \frac{g}{k}} = -k\,dt.$$

Integrating

$$\log\left(v - \frac{g}{k}\right) = -kt + c,$$

or

$$v - \frac{g}{k} = e^c \cdot e^{-kt}.$$

Let $v = v_0$ when $t = 0$, then $e^c = v_0 - \frac{g}{k}$. Therefore

$$v - \frac{g}{k} = \left(v_0 - \frac{g}{k}\right)e^{-kt},$$

or

$$v = \frac{g}{k} + \left(v_0 - \frac{g}{k}\right)e^{-kt}. \qquad (3)$$

Limiting Velocity. — The last equation has a simple interpretation which comes out clearly by plotting the time as abscissa and the velocity as ordinate. There are four special cases which depend upon the following values of the initial velocity:

$$\text{(a)}\ v_0 = 0, \quad \text{(b)}\ v_0 < \frac{g}{k},$$

$$\text{(c)}\ v_0 = \frac{g}{k}, \quad \text{(d)}\ v_0 > \frac{g}{k}.$$

Curves (a), (b), (c), and (d) of Fig. 69 represent these cases. It is evident from these curves that whatever its initial value the velocity tends to the same limiting value $\frac{g}{k}$, called the *limiting velocity*. In the third case the velocity remains constant, as shown by the horizontal line (c), because the resisting force exactly balances the moving force.

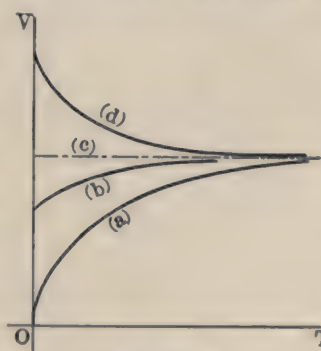


Fig. 69.

Integrating equation (3) we get

$$s = \frac{g}{k}t - \frac{v_0k - g}{k^2}e^{-kt} + c.$$

Let $s = 0$ when $t = 0$, then

$$c = \frac{v_0k - g}{k^2}.$$

Therefore

$$s = \frac{g}{k}t + \frac{v_0k - g}{k^2}(1 - e^{-kt}). \qquad (4)$$

If we plot the last equation for the four different cases of the initial velocity we obtain the curves of Fig. 70. It is evident from these curves that a very short time after the beginning of the motion the distance covered increases at a constant rate, as would be expected from the meaning of the limiting velocity.

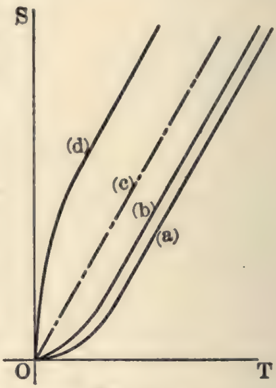


FIG. 70.

CASE II. RESISTANCE PROPORTIONAL TO THE SQUARE OF THE VELOCITY. — The assumption that resistance varies as the velocity holds only for slowly moving bodies. It is found that for projectiles whose velocities lie under 1000 feet per second and over 1500 feet per second the resistance varies, approximately, as the square of the velocity, while between these values it varies as the cube and even higher powers of the velocity. The experimental data on the subject are not enough to find a law of variation which holds in all cases.

If we assume the resistance to vary as the square of the velocity, then the force equation for a falling body becomes

$$m\frac{dv}{dt} = mg - k_1v^2,$$

or

$$\frac{dv}{dt} = g - kv^2, \tag{5}$$

where $k = \dfrac{k_1}{m}$ = constant. In order to integrate the last equation we replace $\dfrac{dv}{dt}$ by $v\dfrac{dv}{ds}$ and rearrange the terms so that we get

$$\frac{v\,dv}{v^2 - \dfrac{g}{k}} = -k\,ds.$$

Integrating we have

$$\log\left(v^2 - \frac{g}{k}\right) = -2\,ks + c,$$

or

$$v^2 - \frac{g}{k} = e^c \cdot e^{-2\,ks}.$$

Let $v = v_0$ when $s = 0$, then $e^c = v_0{}^2 - \frac{g}{k}\cdot$ Therefore

$$v^2 = \frac{g}{k} + \left(v_0{}^2 - \frac{g}{k}\right)e^{-2\,ks}. \tag{6}$$

Therefore the limiting velocity is $\sqrt{\frac{g}{k}}$. In other words, for large values of s the distance traversed approximately equals $\sqrt{\frac{g}{k}}\,t.$

PROBLEMS.

1. A man finds that the resistance of the air to a body moving at the rate of 20 $\frac{\text{km.}}{\text{hr.}}$ equals 1000 dynes per square centimeter of the resisting surface. If 600 $\frac{\text{cm.}}{\text{sec.}}$ is the limit of the velocity with which he can safely land, find the smallest parachute with which he can safely descend from any height. The man and his parachute have a mass of 75 kg. Take the resistance to be proportional to the velocity.

2. In the preceding problem take the resistance to be proportional to the square of the velocity.

3. Discuss the equation of motion of a boat in still water, after the man who was rowing ships his oars. Suppose the resistance to be proportional to the velocity.

4. A particle is projected with a velocity v in a resisting medium and is acted upon by no other force except that due to the resistance of the medium. Show that, (a) the particle will describe a finite distance in an infinite time when the resistance is proportional to the velocity; (b) it will describe infinite distance in infinite time when the resistance is proportional to the square of the velocity.

5. A bullet is projected vertically upwards with a velocity v_0. Supposing the resistance of the air to be proportional to the square of the

velocity of the bullet, find the expression for the highest point reached. Also find the time of upward flight.

6. In the preceding problem suppose the resistance to be proportional to the velocity.

113. IV. Simple Harmonic Motion.—The motion of a particle is called simple harmonic when the particle is acted upon by a force which is always directed towards a fixed point and the magnitude of which is proportional to the distance of the particle from the same point.

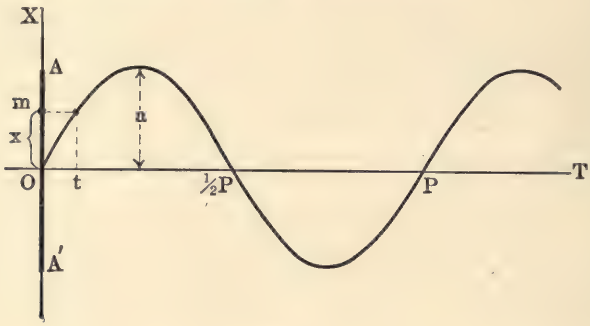


FIG. 71.

Let m be the mass of the particle, the line AA' its path, Fig. 71, and O the fixed point. Then denoting the displacement, i.e., the distance of the particle from the fixed point, by x we obtain

$$\left.\begin{aligned} F &= -k^2x, \\ \text{or} \qquad m\frac{dv}{dt} &= -k^2x, \end{aligned}\right\} \tag{1}$$

for the force equation. It is evident that equation (1) is nothing more or less than the analytical expression for the foregoing definition of simple harmonic motion. The factor k^2 is a constant. The negative sign in equation (1) indicates the fact that the force and the acceleration are directed towards the fixed point, while x is measured from

it. Since the motion is along the x-axis the velocity has no components along the other axes, consequently $v = \frac{dx}{dt}$. Therefore equation (1) may be written in the form

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= -k^2x, \\ \text{or} \qquad \frac{d^2x}{dt^2} &= -\omega^2x, \end{aligned}\right\} \tag{2}$$

where $\omega^2 = \frac{k^2}{m}$. Multiplying both sides of equation (2) by $2\frac{dx}{dt}dt$ and integrating

$$\left(\frac{dx}{dt}\right)^2 = c^2 - \omega^2x^2,$$

or

$$v = \sqrt{c^2 - \omega^2x^2},$$

where c^2 is the constant of integration. Let $v = v_0$ when $x = 0$, then $c^2 = v_0^2$. Therefore

$$v = \sqrt{v_0^2 - \omega^2x^2}. \tag{3}$$

In order to find the second integral of equation (2) rewrite equation (3) in the form

$$\frac{dx}{dt} = \sqrt{v_0^2 - \omega^2x^2}.$$

Separating the variables in the last equation and integrating we have

$$\sin^{-1}\frac{\omega x}{v_0} = \omega t + c',$$

or

$$\begin{aligned} x &= \frac{v_0}{\omega}\sin(\omega t + c') \\ &= a\sin(\omega t + c'), \end{aligned}$$

where c' is the constant of integration and $a = \frac{v_0}{\omega}$. Let $x = 0$ when $t = 0$, then $c' = 0$, and

$$x = a\sin\omega t. \tag{4}$$

When equation (4) is plotted with the time as abscissa and the displacement as ordinate the well-known sine curve is obtained, Fig. 71.

It is evident both from equation (4) and from the curve that the maximum value of x is equal to a. This value of the displacement is called the *amplitude*. The minimum value of x is a displacement equal to the amplitude in the negative direction. Therefore the particle oscillates between the points A and A'. The displacement equals the positive value of the amplitude every time $\sin \omega t$ equals unity, that is, when ωt assumes the values $\frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$, etc. In other words, the particle occupies the extreme point A at the instants when t has the values $\frac{\pi}{2\omega}, \frac{5\pi}{2\omega}, \frac{9\pi}{2\omega}$, etc. Therefore the particle returns to the same point after a lapse of time equal to $\frac{2\pi}{\omega}$. This interval of time is called the *period* of the motion and is denoted by P. Thus

$$P = \frac{2\pi}{\omega}. \tag{5}$$

PROBLEMS.

1. A particle which moves in a straight groove is acted upon by a force which is directed towards a fixed point outside the groove, and which varies as the distance of the particle from the fixed point. Show that the motion is harmonic.

2. Within the earth the gravitational attraction varies as the distance from the center. Find the greatest value of the velocity which a body would attain in falling into a hole, the bottom of which is at the center of the earth.

3. Show that when a particle describes a uniform circular motion, its projection upon a diameter describes a harmonic motion.

GENERAL PROBLEMS.

1. The speed of a train which moves with constant acceleration is doubled in a distance of 3 kilometers. It travels the next $1\frac{1}{10}$ kilometers in one minute. Find the acceleration and the initial velocity.

2. Show that the ratio of the distances described by a falling body during the $(n-1)$th and the nth seconds is $\frac{2n-1}{2n+1}$.

3. A juggler keeps three balls going in one hand, so that at any instant two are in the air and one in his hand. Find the time during which a ball stays in his hand; each ball rises to a height h.

4. Find the shortest time in which a mass m can be raised to a height h by means of a rope which can bear a tension T.

5. A train passes another on a parallel track. When the two locomotives are abreast one of the trains has a velocity of 20 miles per hour and an acceleration of $3\ \frac{\text{ft.}}{\text{sec.}^2}$, while the other has a velocity of 40 miles per hour and an acceleration of $1\ \frac{\text{ft.}}{\text{sec.}^2}$. How soon will they be abreast again, and how far will they have gone in the meantime?

6. A mass of 1 kg. is hanging from a spring balance in an elevator. After the elevator starts the balance reads 1100 gm. Assuming the acceleration of the elevator to be constant, find the distance moved in 5 seconds.

7. A smooth inclined plane of mass m and inclination α stands with its base on a smooth horizontal plane. What horizontal force must be applied to the plane in order that a particle placed on the plane simultaneously with the beginning of action of the force may be in contact with the plane yet fall vertically down as if the inclined plane were not there?

8. The pull of a train exceeds the resisting forces by 0.02 of the weight of the train. When the brakes are on full the resisting forces equal 0.1 of the weight. Find the least time in which the train can travel between two stopping stations 5 miles apart, the tracks being level.

9. Give a construction for finding the line of quickest descent from a point to a circle in the same vertical plane.

10. A mass m_1 falling vertically draws a mass m_2 up a smooth inclined plane which makes an angle of 30° with the horizon. The masses are connected with a string which passes over a small smooth pulley at the top of the plane. Find the ratio of the masses which will make the acceleration $\frac{g}{4}$.

11. A particle is projected up an inclined plane which makes an angle α with the horizon. If T_1 is the time of ascent, T_2 the time of descent, and ϕ the angle of friction, show that

$$\left(\frac{T_1}{T_2}\right)^2 = \frac{\sin(\alpha-\phi)}{\sin(\alpha+\phi)}.$$

12. The time of descent along straight lines from a point on a vertical circle to the center and to the lowest point is the same. Find the position of the point.

13. A uniform cord of mass m and length l passes over a smooth peg and hangs vertically. If it slides freely, show that the tension of the cord equals $\frac{4x(l-x)}{l^2}$, when the length on one side is x.

14. In an Atwood's machine experiment the sum of the two moving masses is m. Find their values if in t seconds they move through a distance h.

15. Given the height h of an inclined plane, show that its length must be $\frac{2m_2}{m_1 h}$, in order that m_1, descending vertically, shall draw m_2 up the plane in the least possible time.

16. A gun points at a target suspended from a balloon. Show that if the target be dropped at the instant the gun is discharged, the bullet will hit the target if the latter is within the bullet's range.

17. Find the position where a particle sliding along the outside of a smooth vertical circle will leave the circle.

18. A particle falls towards a fixed point under the action of a force which equals $\gamma r^{-\frac{5}{3}}$, where γ is a constant and r is the distance of the particle from the fixed point. Show that starting from a distance a the particle will arrive at the fixed point with an infinite velocity in the time $\frac{2a^{\frac{4}{3}}}{\sqrt{3\gamma}}$.

19. A particle falls towards a fixed point under the attraction of a force which varies with some power of the distance of the particle from the center of attraction. Find the law of force, supposing the velocity acquired by the particle in falling from an infinite distance to a distance a from the center to be equal to the velocity acquired in falling from rest from a distance a to a distance $\frac{a}{4}$.

20. A particle is projected toward a center of attraction with a velocity equal to the velocity it would have acquired had it fallen from an infinite distance to the position of projection. Supposing the force of attraction to be γr^{-n}, where γ is a constant and r is the distance of the particle from the center of attraction, show that the time taken to cover the distance between the point of projection and the center of attraction is

$$\frac{2}{n+1}\sqrt{\frac{n-1}{2\gamma}}\cdot a^{\frac{n+1}{2}}.$$

21. From the following data show that the velocity with which a body has to be projected from the moon in order to reach the earth is about 1.5 miles per sec. Both the earth and the moon are supposed to be at rest. The mass of the moon is $\frac{1}{81}$ of that of the earth. The radii of the earth and the moon are 4000 miles and 1100 miles, respectively. The distance between the earth and the moon is 240,000 miles.

22. Taking the data of the preceding problem, show that if the earth and the moon were reduced to rest they would meet, under their mutual attraction, in about 4.5 days.

CHAPTER VII.

CENTER OF MASS AND MOMENT OF INERTIA.

CENTER OF MASS.

THERE are two useful conceptions, known as center of mass and moment of inertia, which greatly simplify discussions of the motion of rigid bodies. It is, therefore, desirable to become familiar with these conceptions before taking up the motion of rigid bodies.

114. Definition of Center of Mass. — The center of mass of a system of equal particles is their average position; in other words, it is that point whose distance from any fixed plane is the average of the distances of all the particles of the system.

Let $x_1, x_2, x_3, \ldots x_n$ denote the distances of the particles of a system from the *yz*-plane; then, by the above definition, the distance of the center of mass from the same plane is

$$\begin{aligned} \bar{x} &= \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} \\ &= \frac{1}{n}\Sigma x. \end{aligned}$$

When the particles have different masses their distances must be weighted, that is, the distance of each particle must be multiplied by the mass of the particle before taking the average. In this case the distance of the center of mass from the *yz*-plane is defined by the following equation:

$$(m_1 + m_2 + \cdots + m_n)\,\bar{x} = m_1 x_1 + m_2 x_2 + \cdots + m_n x_n,$$

$$\left.\begin{aligned} &\text{or} && \bar{x} = \frac{\Sigma mx}{\Sigma m}, \\ &\text{similarly} && \bar{y} = \frac{\Sigma my}{\Sigma m}, \\ &\text{and} && \bar{z} = \frac{\Sigma mz}{\Sigma m}. \end{aligned}\right\} \qquad (\text{I}')$$

Evidently $\bar{x}$, $\bar{y}$, and $\bar{z}$ are the coördinates of the center of mass.

ILLUSTRATIVE EXAMPLES.

1. Find the center of mass of two particles of masses m and nm, which are separated by a distance a.

Taking the origin of the axes at the particle which has the mass m, Fig. 72, and taking as the x-axis the line which joins the two particles we get

$$\begin{aligned} \bar{x} &= \frac{0 + nma}{m + nm} \\ &= \frac{n}{n+1}a, \\ \bar{y} &= 0, \\ \bar{z} &= 0. \end{aligned}$$

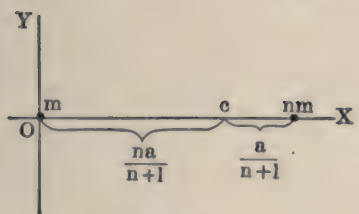


Fig. 72.

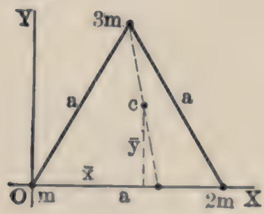


Fig. 73.

2. Find the center of mass of three particles of masses m, $2\,m$, and $3\,m$, which are at the vertices of an equilateral triangle of sides a.

Choosing the axes as shown in Fig. 73 we have

$$\begin{aligned} \bar{x} &= \frac{0 + 2\,ma + 3\,ma\cos 60^\circ}{m + 2\,m + 3\,m} \\ &= \tfrac{7}{12}\,a, \\ \bar{y} &= \frac{0 + 0 + 3\,ma\sin 60^\circ}{6\,m} \\ &= \tfrac{1}{4}\sqrt{3}\,a, \\ \bar{z} &= 0. \end{aligned}$$

115. Center of Mass of Continuous Bodies.—When the particles form a continuous body we can replace the summation signs of equation (I′) by integration signs and obtain the following expressions for the coördinates of the center of mass:

$$\left.\begin{aligned} \bar{x} &= \frac{\int_0^m x\,dm}{\int_0^m dm}, * \\ \bar{y} &= \frac{\int_0^m y\,dm}{\int_0^m dm}, \\ \bar{z} &= \frac{\int_0^m z\,dm}{\int_0^m dm}, \end{aligned}\right\} \qquad \text{(I)}$$

where m is the mass of the body.

ILLUSTRATIVE EXAMPLES.

1. Find the center of mass of the parabolic lamina bounded by the curves $y^2 = 2\,px$ and $x = a$, Fig. 74.

Obviously the center of mass lies on the x-axis. Therefore we need to

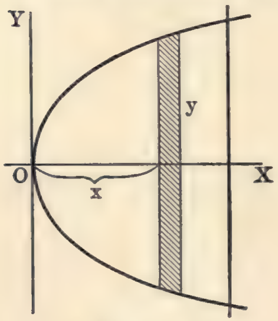


FIG. 74.

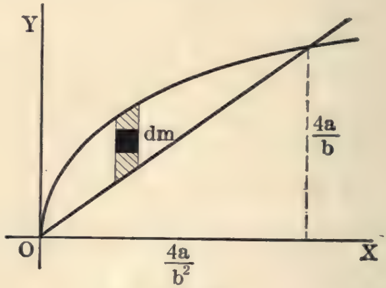


FIG. 75.

* In general if y is a function of x then the average value of y between the limits x_1 and x_2 is given by the relation: $\bar{y} = \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} y\,dx.$

determine $\bar{x}$ only. Taking a strip of width dx for the element of mass we have

$$\begin{aligned} dm &= \sigma\, 2\, y\, dx \\ &= 2\, \sigma \sqrt{2\, px}\, dx, \end{aligned}$$

where σ is the mass per unit area. Therefore substituting this expression of dm in equation (I) and changing the limits of integration we obtain

$$\begin{aligned} \bar{x} &= \frac{2\, \sigma \int_0^a x \sqrt{2\, px}\, dx}{2\, \sigma \int_0^a \sqrt{2\, px}\, dx} \\ &= \frac{\int_0^a x^{\frac{3}{2}}\, dx}{\int_0^a x^{\frac{1}{2}}\, dx} \\ &= \frac{3\, a}{5}. \end{aligned}$$

2. Find the center of mass of the lamina bounded by the curves $y^2 = 4\, ax$ and $y = bx$, Fig. 75.

Let $dx\, dy$ be the area of the element of mass, then

$$dm = \sigma\, dx\, dy.$$

Therefore substituting in equation (I) and introducing the proper limits of integration we obtain

$$\begin{aligned} \bar{x} &= \frac{\int_0^{\frac{4\,a}{b^2}} \int_{bx}^{2\sqrt{ax}} x\, dy\, dx}{\int_0^{\frac{4\,x}{b^2}} \int_{bx}^{2\sqrt{ax}} dy\, dx} \\ &= \frac{\int_0^{\frac{4\,a}{b^2}} (2\sqrt{ax} - bx)\, x\, dx}{\int_0^{\frac{4\,a}{b^2}} (2\sqrt{ax} - bx)\, dx} \\ &= \frac{8\, a}{5\, b^2}. \end{aligned}$$

$$\begin{aligned} \bar{y} &= \frac{\int_0^{\frac{4\,a}{b^2}} \int_{bx}^{2\sqrt{ax}} y\, dy\, dx}{\int_0^{\frac{4\,a}{b^2}} \int_{bx}^{2\sqrt{ax}} dy\, dx} \\ &= \frac{\int_0^{\frac{4\,a}{b^2}} \left(2\, ax - \frac{b^2}{2} x^2\right) dx}{\int_0^{\frac{4\,a}{b^2}} (2\sqrt{ax} - bx)\, dx} \\ &= \frac{2\, a}{b}. \end{aligned}$$

3. Find the center of mass of a semicircular lamina.

Selecting the coördinates and the element of mass as shown in Fig. 76 we have

$$dm = \sigma \cdot \rho \, d\theta \cdot d\rho,$$

$$\bar{y} = \frac{\int_0^\pi \int_0^a y \cdot \sigma\rho \, d\rho \, d\theta}{\int_0^\pi \int_0^a \sigma\rho \, d\rho \, d\theta}$$

$$= \frac{\int_0^\pi \int_0^a \rho^2 \sin\theta \, d\rho \, d\theta}{\int_0^\pi \int_0^a \rho \, d\rho \, d\theta}$$

$$= \frac{4a}{3\pi},$$

$$\bar{x} = 0.$$

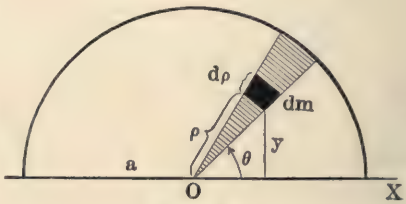


FIG. 76.

PROBLEMS.

Find the center of mass of the lamina bounded by the following curves:

(1) $y = mx$, $y = -mx$, and $y = a$.
(2) $y = a \sin x$, $y = 0$, $x = 0$, and $x = \pi$.
(3) $y^2 = ax$ and $x^2 = by$.
(4) $x^2 + y^2 = a^2$, $x = 0$, and $y = 0$.
(5) $b^2x^2 + a^2y^2 = a^2b^2$, $x = 0$, and $y = 0$.
(6) $r = a(1 + \cos\theta)$.
(7) $r = a$, $\theta = 0$, and $\theta = \theta$.
(8) $r = a$, $r = b$, $\theta = 0$, and $\theta = \frac{\pi}{4}$.

116. Center of Mass of a Homogeneous Solid of Revolution. — Let Fig. 77 represent any solid obtained by revolving a plane curve about the x-axis. Then the center of mass lies on the axis of revolution. The position of the center of mass is found most conveniently when the element of mass is a thin slice obtained by two transverse sections. The expression for the mass of such an element is

$$dm = \tau \cdot \pi y^2 \cdot dx,$$

where τ is the density of the solid, y the radius of the slice, and dx its thickness.

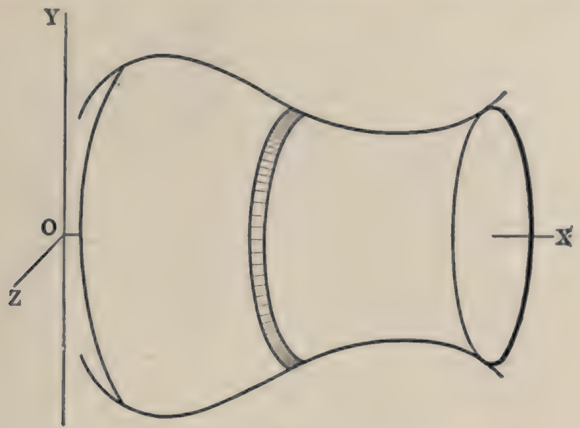


FIG. 77.

ILLUSTRATIVE EXAMPLE.

Find the center of mass of a paraboloid of revolution obtained by revolving about the x-axis that part of the parabola $y^2 = 2\,px$ which lies between the lines $x = 0$ and $x = a$.

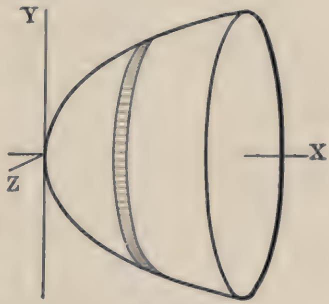


FIG. 78.

$$dm = \tau\pi y^2\,dx$$
$$= \tau\pi\, 2\,px\,dx;$$
$$\therefore\quad \bar{x} = \frac{2\,\pi\tau p\int_0^a x^2\,dx}{2\,\pi\tau p\int_0^a x\,dx}$$
$$= \tfrac{2}{3}\,a.$$

PROBLEMS.

Find the center of mass of the homogeneous solid of revolution generated by revolving about the x-axis the area bounded by

(1) $y = \dfrac{a}{h}x$, $x = h$, and $y = 0$.

(2) $x^2 = 4\,ay$, $x = 0$, and $y = a$.

(3) $x^2 + y^2 = a^2$, and $x = 0$.

(4) $b^2x^2 + a^2y^2 = a^2b^2$, and $x = 0$.

(5) $y = \sin x$, $x = 0$, and $x = \dfrac{\pi}{2}$.

(6) $x^2 + y^2 = a^2$, $x^2 + y^2 = b^2$, and $x = 0$.

117. Center of Mass of Filaments. — The transverse dimensions of a filament are supposed to be negligible; therefore it can be treated as a geometrical curve. Taking a piece of length ds as the element of mass and denoting the mass per unit length by λ we have

$$dm = \lambda\, ds.$$

ILLUSTRATIVE EXAMPLE.

Find the center of mass of a semicircular filament.

(a) Taking $x^2 + y^2 = a^2$ to be the equation of the circle we get

$$\begin{aligned} dm &= \lambda\, ds \\ &= \lambda \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \lambda\, a \frac{dx}{\sqrt{a^2 - x^2}}. \end{aligned}$$

$$\therefore \quad \bar{x} = \frac{\displaystyle\int_0^a x \frac{dx}{\sqrt{a^2 - x^2}}}{\displaystyle\int_0^a \frac{dx}{\sqrt{a^2 - x^2}}} = \left[\frac{-\sqrt{a^2 - x^2}}{\sin^{-1}\dfrac{x}{a}}\right]_0^a = \frac{2\,a}{\pi}.$$

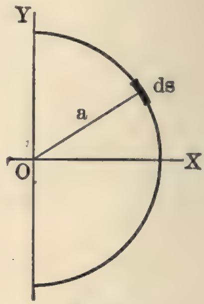


FIG. 79.

(b) Referring the circle to polar coördinates we have $r = a$ for its equation. Therefore

$$\begin{aligned} dm &= \lambda\, ds \\ &= \lambda\, a\, d\theta. \end{aligned}$$

$$\therefore \quad \bar{x} = \frac{\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\, \lambda a\, d\theta}{\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \lambda\, a\, d\theta} = \frac{\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a \cos\theta\, d\theta}{\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta} = \frac{2\,a}{\pi}.$$

PROBLEMS.

Find the center of mass of a uniform wire bent into the following curves:

(1) An arc of a circle subtending an angle 2θ at the center.

(2) $y = a \sin x$, between $x = 0$ and $x = \pi$.

(3) $y^2 = 4ax$, between $x = 0$ and $x = 2a$.

(4) The cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, between two successive cusps.

(5) Half of the cardioid $r = a(1 + \cos\theta)$.

118. Center of Mass of a Body of Any Shape and Distribution of Mass. — The illustrative examples of the last few pages are worked out by special methods in order to bring out the fact that in a great number of problems the ease with which the center of mass may be determined depends upon the choice of the element of mass. The following general expressions for an element of mass may be used whatever the shape of the body or the distribution of its mass:

(a) When the bounding surfaces of the body are given in the Cartesian coördinates the mass of an infinitesimal cube is taken as the element of mass:

$$dm = \tau \cdot dx\,dy\,dz.$$

(b) When the bounding surfaces of the body are given in spherical coördinates the element of mass is chosen as shown in Fig. 80. In this case the following is the expression for the element of mass:

$$dm = \tau \cdot \rho\,d\theta \cdot d\rho \cdot \rho\,d\phi \sin\theta = \tau\rho^2 \sin\theta\,d\theta\,d\phi\,d\rho.$$

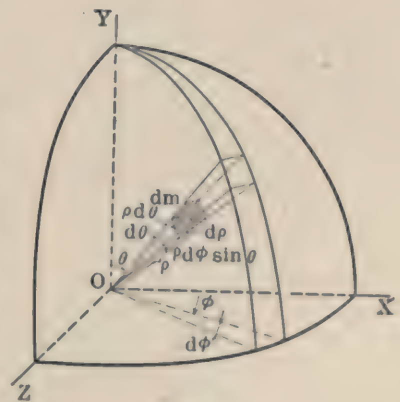


FIG. 80.

(c) When the density, τ, varies from point to point it is expressed in terms of the coördinates and substituted in the expression for dm.

ILLUSTRATIVE EXAMPLES.

1. Find the center of mass of an octant of a homogeneous sphere.

(a) Suppose the bounding surfaces to be

$$x^2 + y^2 + z^2 = a^2,\ x = 0,\ y = 0, \text{ and } z = 0.$$

Then the limits of integration are

$$x = 0 \text{ and } x = a,$$

$$y = 0 \text{ and } y = \sqrt{a^2 - x^2},$$

$$z = 0 \text{ and } z = \sqrt{a^2 - x^2 - y^2}.$$

Therefore
$$\bar{x} = \frac{\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} x\,dx\,dy\,dz}{\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dx\,dy\,dz}$$

$$= \frac{3a}{8},$$

and by symmetry $\bar{y} = \bar{z} = \frac{3a}{8}\cdot$

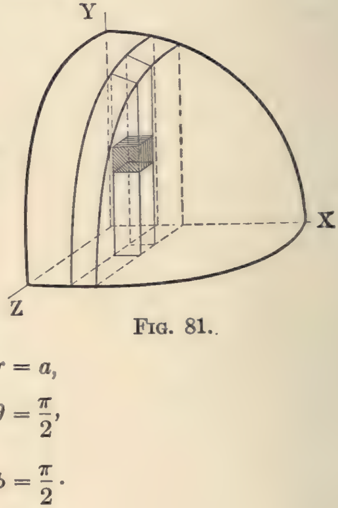


FIG. 81.

(b) Suppose the equations of the bounding surfaces to be given in spherical coördinates, then we have

$$r = a,\ \theta = \frac{\pi}{2},\ \phi = 0, \text{ and } \phi = \frac{\pi}{2}\cdot$$

The limits of integration are

$$r = 0 \quad \text{and} \quad r = a,$$

$$\theta = 0 \quad \text{and} \quad \theta = \frac{\pi}{2},$$

$$\phi = 0 \quad \text{and} \quad \phi = \frac{\pi}{2}\cdot$$

Therefore
$$\bar{x} = \frac{\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^a r^3 \sin^2\theta \cos\phi\,dr\,d\theta\,d\phi}{\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^a r^2 \sin\theta\,dr\,d\theta\,d\phi} \qquad [x = r\sin\theta\cos\phi]$$

$$= \frac{3a}{8}.$$

2. Find the center of mass of a right circular cone whose density varies inversely as the square of the distance from the apex, the distance being measured along the axis.

$$dm = \tau \cdot \pi y^2 \cdot dx$$

$$= \frac{\tau_1}{x^2} \cdot \pi \frac{a^2 x^2}{h^2} \cdot dx \qquad \left[\frac{y}{x} = \frac{a}{h}\right]$$

$$= \frac{\tau_1 \pi a^2}{h^2}\, dx,$$

where τ_1 is the density at a unit distance from the apex. Therefore

$$\bar{x} = \frac{\dfrac{\tau_1 \pi a^2}{h^2} \displaystyle\int_0^h x\, dx}{\dfrac{\tau_1 \pi a^2}{h^2} \displaystyle\int_0^h dx}$$

$$= \frac{h}{2}.$$

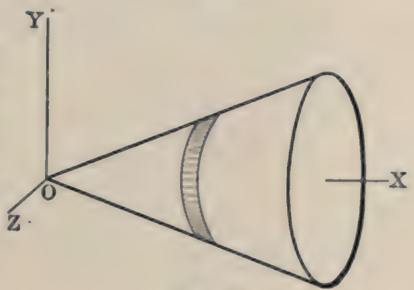


FIG. 82.

PROBLEMS.

1. Find the center of mass of a right circular cone, the density of which varies inversely as the distance from the vertex.

2. Find the center of mass of a circular plate, the density of which varies as the distance from a point on the circumference.

3. Find the center of mass of a cylinder, the density of which varies with the nth power of the distance from one base.

4. Find the center of mass of a quadrant of an ellipsoid.

5. Find the center of mass of a hemisphere, the density of which varies as the distance from the center.

119. Center of Mass of a Number of Bodies. — Let m_1, m_2, etc., be the masses and $\bar{x}_1$, $\bar{x}_2$, etc., be the x-coördinates of the centers of mass of the individual bodies. Then if $\bar{x}$ denotes the x-coördinate of the center of mass of the entire system we can write

$$x = \frac{\int x\, dm}{\int dm}$$

$$= \frac{\int_0^{m_1} x\,dm + \int_0^{m_2} x\,dm + \cdots}{\int_0^{m_1} dm + \int_0^{m_2} dm + \cdots}$$

$$= \frac{m_1\bar{x}_1 + m_2\bar{x}_2 + \cdots}{m_1 + m_2 + \cdots}$$

$$= \frac{\Sigma m\bar{x}_i}{\Sigma m}.$$

Similarly $$y = \frac{\Sigma m\bar{y}_i}{\Sigma m},$$

$$\bar{z} = \frac{\Sigma m\bar{z}_i}{\Sigma m}.$$

Therefore the mass of each body may be considered as being concentrated at its center of mass.

ILLUSTRATIVE EXAMPLE.

Find the center of mass of the plate indicated by the shaded part of Fig. 83.

(a) Suppose the plate to be separated into two parts by the dotted line. Then the coördinates of the center of mass of the lower part are

$$\bar{x}_1 = \frac{b}{2} \quad \text{and} \quad \bar{y}_1 = \frac{b-a}{2}.$$

On the other hand the coördinates of the center of mass of the upper part are

$$\bar{x}_2 = \frac{b-a}{2} \quad \text{and} \quad \bar{y}_2 = \frac{2b-a}{2}.$$

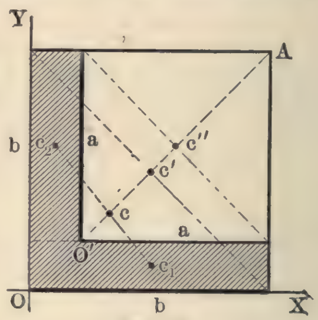


FIG. 83.

Therefore the coördinates of the center of mass of the entire plate have the following values:

$$\bar{x} = \frac{m_1\frac{b}{2} + m_2\frac{b-a}{2}}{m_1 + m_2} = \frac{\sigma b(b-a)\frac{b}{2} + \sigma a(b-a)\frac{b-a}{2}}{\sigma b\,(b-a) + \sigma\,(b-a)\,a} = \frac{b^2 + ab - a^2}{2\,(a+b)}.$$

$$\bar{y} = \frac{m_1\frac{b-a}{2} + m_2\frac{2b-a}{2}}{m_1 + m_2} = \frac{\sigma b(b-a)\frac{b-a}{2} + \sigma a(b-a)\frac{2b-a}{2}}{\sigma b\,(b-a) + \sigma a\,(b-a)} = \frac{b^2 + ab - a^2}{2\,(a+b)}.$$

(b) Suppose the square OA to represent a plate of positive mass and the square $O'A$ to represent a plate of negative mass. Then if the two plates have the same thickness and density the positive and the negative masses annul each other in the square $O'A$. Therefore the two square plates form a system which is equivalent to the actual plate represented by the shaded area of the figure. Hence the center of mass of the square plates is also the center of mass of the given plate.

The masses of the square plates are σb^2 and $-\sigma a^2$, while the coördinates of their centers of mass are

$$\bar{x}' = \bar{y}' = \frac{b}{2} \quad \text{and} \quad \bar{x}'' = \bar{y}'' = \frac{2b-a}{2}.$$

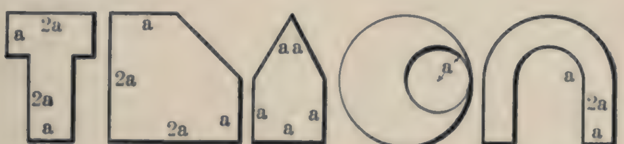
Therefore the coördinates of the center of mass of the two are

$$\bar{x} = \bar{y} = \frac{\sigma b^2 \frac{b}{2} + (-\sigma a^2)\frac{2b-a}{2}}{\sigma b^2 + (-\sigma a^2)} = \frac{b^2 + ab - a^2}{2(a+b)},$$

which are identical with those obtained by the first method.

PROBLEMS.

1. Find the center of mass of the homogeneous plates indicated by the following figures:



(a) (b) (c) (d) (e)

2. A sphere of radius b has a spherical cavity of radius a. Find the center of mass if the distance between the centers is c.

3. A right cone is cut from a right circular cylinder of the same base and altitude. Find the center of mass of the remaining solid.

4. A right cone is cut from a hemisphere of the same base and altitude. Find the center of mass of the remaining solid.

5. A right circular cone is cut from another right circular cone of the same base but of greater altitude. Find the center of mass of the remaining solid.

6. A right circular cone is cut from the paraboloid of revolution gener-

ated by revolving about the x-axis the area bounded by $y^2 = 2\,px$ and $x = a$. Find the center of mass of the remaining solid if the paraboloid and the cone have the same base and vertex.

MOMENT OF INERTIA.

120. Definition of Moment of Inertia. — The moment of inertia of a body about an axis equals the sum of the products of the masses of the particles of the body by the square of their distances from the axis.* Thus if dm denotes an element of mass of the body and r its distance from the axis then the following is the analytical statement of the definition of moment of inertia:

$$I = \int_0^m r^2\,dm. \qquad \text{(II)}$$

The integration which is involved in equation (II) is often simplified by a proper choice of the element of mass. The choice depends upon the bounding surfaces of the body and the position of the axis; therefore there is no general rule by which the most convenient element of mass may be selected. There is one important point, however, which the student should always keep in mind in selecting the element of mass, namely, *the distances of the various parts of the element of mass from the axis must not differ by more than infinitesimal lengths.*

ILLUSTRATIVE EXAMPLES.

1. Find the moment of inertia of a rectangular lamina about one of its sides.

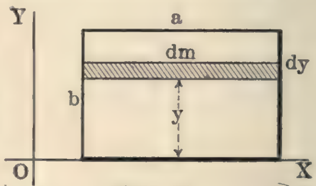


FIG. 84.

Suppose the lamina to lie in the xy-plane. Further suppose the side with respect to which the moment of inertia is to be found to lie in the x-axis. Then the most convenient element of mass is a strip which is parallel to the x-axis.

Let a be the length (Fig. 84), b the width, and σ the mass per unit area of the lamina, then

$$dm = \sigma a\,dy.$$

* For a physical definition of moment of inertia and its meaning see p. 220.

The distance of the element of mass from the axis is y; therefore substituting in equation (II) these expressions for dm and its distance from the axis we obtain

$$I = \int_0^b y^2 \cdot \sigma a \, dy$$

$$= \tfrac{1}{3} \sigma a b^3$$

$$= \tfrac{1}{3} m b^2,$$

for the desired moment of inertia. The limits of integration are different from those in equation (II) because the independent variable is changed from m to y.

2. Find the moment of inertia about the x-axis of a lamina which is bounded by the parabola $x^2 = 2\,py$ and the straight line $y = a$.

(a) Choosing a horizontal strip for the element of mass we have

$$dm = \sigma \cdot 2\,x\,dy.$$

$$\therefore \quad I = 2\,\sigma \int_0^a y^2 x \, dy$$

$$= 2\,\sigma \int_0^a y^2 \sqrt{2\,py}\, dy$$

$$= \tfrac{4}{7}\, \sigma\, a^3 \sqrt{2\,pa}.$$

But

$$m = \sigma \int_0^a 2\,x\,dy$$

$$= \tfrac{4}{3}\, \sigma\, a \sqrt{2\,pa}.$$

$$\therefore \quad I = \tfrac{3}{7}\, m a^2.$$

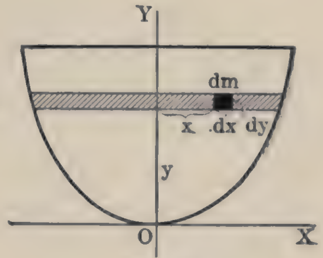


Fig. 85.

(b) We can also take an element of the strip for the element of mass, in which case we have

$$dm = \sigma \, dx \, dy.$$

$$\therefore \quad I = \int_0^a \int_{-\sqrt{2\,py}}^{\sqrt{2\,py}} y^2 \cdot \sigma \, dx \, dy$$

$$= \tfrac{4}{7}\, \sigma\, a^3 \sqrt{2\,pa},$$

$$= \tfrac{3}{7}\, m a^2.$$

PROBLEMS.

1. Find the moment of inertia of a circular lamina about a diameter.

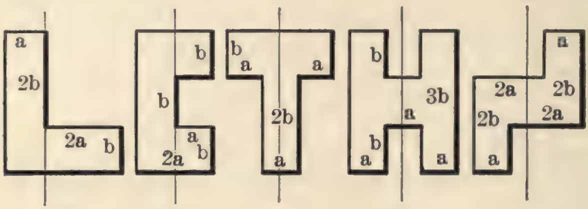
2. Find the moment of inertia of an elliptical lamina about its minor axis.

3. Find the moment of inertia of a rectangular plate of negligible thickness about a diagonal.

4. Find the moment of inertia of a thin plate, which is in the shape of an equilateral triangle, with respect to one of its edges.

5. Find the moment of inertia of a triangular plate about an axis which passes through one of its vertices and is parallel to the base.

6. Find the moments of inertia of the following laminæ with respect to the axes indicated by the thin vertical lines.



(a) (b) (c) (d) (e)

121. Theorems on Moments of Inertia. Theorem I. — *The moment of inertia of a lamina about an axis which is perpendicular to its plane equals the sum of the moments of inertia with respect to two rectangular axes which lie in the plane of the lamina with their origin on the first axis.*

Suppose the lamina to be in the xy-plane, then the theorem states that the moment of inertia about the z-axis equals the sum of the moments of inertia about the other two axes, that is,

$$I_z = I_x + I_y. \qquad \text{(III)}$$

The following analysis explains itself.

$$\begin{aligned} I_z &= \int_0^m r^2\, dm \\ &= \int_0^m (x^2 + y^2)\, dm \\ &= \int_0^m x^2\, dm + \int_0^m y^2\, dm \\ &= I_x + I_y. \end{aligned}$$

It is evident from this theorem that when the lamina is rotated about the z-axis I_x and I_y change, in general, but their sum remains constant.

122. Theorem II. — *The moment of inertia of a body about any axis equals its moment of inertia about a parallel axis through the center of mass plus the product of the mass of the body by the square of the distance between the two axes.*

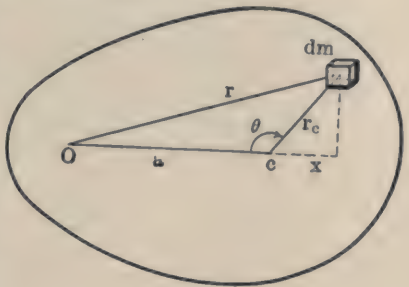


Fig. 86.

Let the axis be perpendicular to the plane of the paper and pass through the point O, Fig. 86. Further let dm be any element of mass, r its distance from the axis through O, and r_c its distance from a parallel axis through the center of mass, C. Then if a denotes the distance between the axes we have

$$\begin{aligned} I &= \int_0^m r^2\,dm \\ &= \int_0^m (r_c^2 + a^2 - 2\,ar_c \cos\theta)\,dm \\ &= \int_0^m r_c^2\,dm + \int_0^m a^2\,dm - 2\,a\int_0^m r_c \cos\theta\,dm \\ &= I_c + ma^2 - 2\,a\int_0^m x\,dm. \end{aligned}$$

But by the definition of the center of mass $\int_0^m x\,dm = m\bar{x}$, and in the present case the center of mass is at the origin; therefore $\bar{x}$ and consequently the last integral vanishes. Thus we get

$$I = I_c + ma^2. \qquad \text{(IV)}$$

123. Radius of Gyration.—The radius of gyration of a body with respect to an axis is defined as the distance from the

axis of a point where if all the mass of the body were concentrated its moment of inertia would not change.

Let m denote the mass of a body, I its moment of inertia with respect to a given axis, and K its radius of gyration relative to the same axis; then the definition gives

$$\left.\begin{aligned} I &= K^2 m, \\ \text{or} \qquad K &= \sqrt{\frac{I}{m}}. \end{aligned}\right\} \tag{V}$$

If K_c denote the radius of gyration relative to a parallel axis through the center of mass, then by equations (IV) and (V) we obtain

$$K^2 = K_c^{\,2} + a^2. \tag{VI}$$

ILLUSTRATIVE EXAMPLES.

1. Find the moment of inertia of a homogeneous circular disk (a) about its geometrical axis, (b) about one of the elements of its lateral surface.

Let m be the mass, a the radius, l the thickness, and τ the density of the disk. Then choosing a circular ring for the element of mass we have

$$dm = \tau \cdot l \cdot 2\,\pi r \cdot dr,$$

where r is the radius of the ring and dr its thickness. Therefore the moment of inertia about the axis of the disk is

$$\begin{aligned} I &= 2\,\pi l \tau \int_0^a r^3\,dr \\ &= \frac{\tau l \pi a^4}{2} \\ &= \frac{ma^2}{2}. \end{aligned}$$

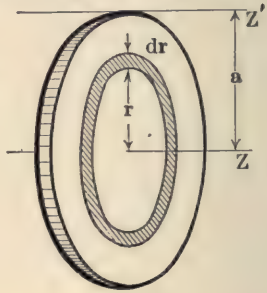


Fig. 87.

The moment of inertia about the element is obtained easily by the help of theorem II. Thus

$$\begin{aligned} I' &= I + ma^2 \\ &= \tfrac{3}{2}\,ma^2. \end{aligned}$$

It will be noticed that the thickness of the disk does not enter into the expressions for I and I' except through the mass of the disk. Therefore these expressions hold good whether the disk is thick enough to be called a cylinder or thin enough to be called a circular lamina.

2. Find the moment of inertia of a cylinder about a transverse axis through the center of mass.

Let m, a, l, and τ be, respectively, the mass, the radius, the length, and the density of the cylinder. Further let the given axis pass through the center of mass of the cylinder; then taking a slice obtained by two right sections as the element of mass we get, by theorem II,

$$dI_y = dI_{y'} + z^2\,dm,$$

where dm is the mass of the element, dI_y and $dI_{y'}$ are the moments of inertia of the element about the given axis and about a parallel axis through the center of mass of the element, and z is the distance between these two axes. But by theorem I

$$dI_{y'} + dI_{x'} = dI_z,$$

and by symmetry $$dI_{x'} = dI_{y'},$$

and by the last illustrative example

$$dI_z = \frac{a^2\,dm}{2}.$$

Therefore $$dI_{y'} = \frac{a^2\,dm}{4}.$$

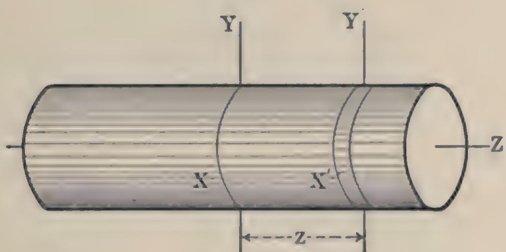


FIG. 88.

Substituting this value of $dI_{y'}$ in the expression for dI_y we get

$$dI_y = \left(\frac{a^2}{4} + z^2\right)dm$$

Integrating the last equation we have

$$I_y = \frac{ma^2}{4} + \int_0^m z^2\,dm$$

$$= \frac{ma^2}{4} + \int_{-\frac{l}{2}}^{\frac{l}{2}} z^2 \cdot \tau\pi a^2\,dz$$

$$= \frac{ma^2}{4} + \frac{\tau\pi a^2 l^3}{12}$$

$$= m\left(\frac{a^2}{4} + \frac{l^2}{12}\right).$$

3. Find the moment of inertia of a homogeneous sphere about a diameter and about a tangent line.

Let m, a, and τ be the mass, the radius, and the density of the sphere, respectively. Then, taking the axes and the element of mass as shown in Fig. 89, we have

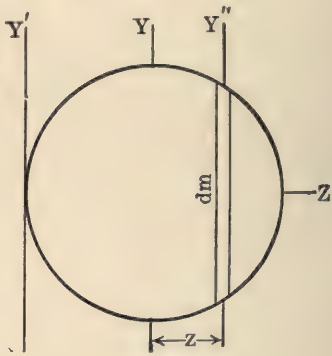


FIG. 89.

$$dI_y = dI_{y''} + z^2\, dm,$$

$$= \frac{1}{2} dI_z + z^2\, dm$$

$$= \frac{1}{2}\frac{y^2\, dm}{2} + z^2\, dm.$$

Integrating the last equation

$$I_y = \tfrac{1}{4}\int_0^m y^2\, dm + \int_0^m z^2\, dm$$

$$= \frac{\tau\pi}{4}\int_{-a}^{a} y^4\, dz + \tau\pi\int_{-a}^{a} z^2 y^2\, dz \qquad [dm = \tau\pi y^2 dz]$$

$$= \frac{\tau\pi}{2}\int_0^a (a^2 - z^2)^2\, dz + 2\,\tau\pi\int_0^a (a^2 - z^2)\, z^2 dz$$

$$= \frac{8\,\tau\pi a^5}{15}$$

$$= \tfrac{2}{5}\, ma^2$$

and
$$I_{y'} = I_y + ma^2$$
$$= \tfrac{7}{5}\, ma^2.$$

124. Theorem III. — *The moment of inertia of a homogeneous right cylinder about a transverse axis equals the moment of inertia of two laminæ which fulfill the following conditions.* (*a*) *Each lamina has a mass equal to that of the cylinder.* (*b*) *One lamina occupies the entire area of the transverse section of the cylinder through the given axis, while the other lamina occupies the entire area of the longitudinal section of the cylinder through the axis.*

Let Y, Fig. 90, be the axis with respect to which it is desired to find the moment of inertia of the cylinder. Let dI_y denote the moment of inertia of an element bounded by two transverse sections relative to the Y-axis, and $dI_{y''}$ denote the moment of inertia of the same element relative to the

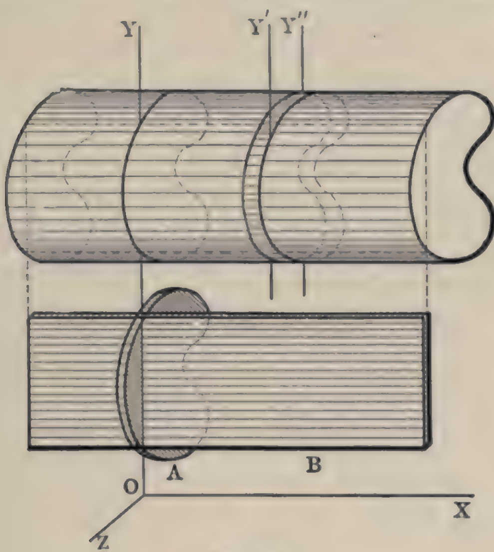


FIG. 90.

Y''-axis, a parallel axis through the center of mass of the element. Then, by theorem II, we have

$$dI_y = dI_{y''} + (x^2 + z^2)\,dm,$$

where $(x^2 + z^2)$ is the square of the distance between the two axes. Similarly the moment of inertia of the element about the Y'-axis which is parallel to the Y-axis and intersects the same elements of the cylinder, is given by

$$dI_{y'} = dI_{y''} + z^2\,dm.$$

Eliminating $dI_{y''}$ between the last two equations we obtain

$$\begin{aligned} dI_y &= dI_{y'} + x^2\,dm \\ &= K_1^2\,dm + x^2\,dm, \end{aligned}$$

where K_1 is the radius of gyration of the element of mass about the Y'-axis. Integrating the last equation we have

$$I_y = \int_0^m K_1^2\, dm + \int_0^m x^2\, dm.$$

Each of the elements of mass has its own Y'-axis similarly placed. Therefore k_1 is the same for all the elements of mass and remains constant during the integration. Hence

$$\begin{aligned} I_y &= K_1^2 m + \int_0^m x^2\, dm \\ &= I_1 + I_2, \end{aligned}$$

where $I_1 = K_1^2 m$ and $I_2 = \int_0^m x^2\, dm$. It is not difficult to see that I_1 is the sum of the moments of inertia of all the elements of mass relative to their respective Y'-axes. It is equal, therefore, to the moment of inertia about the Y-axis of the lamina (A in the figure) which would be obtained if the entire cylinder could be compressed into the transverse section through the Y-axis. On the other hand I_2 equals the moment of inertia about the Y-axis of the lamina (B in the figure) which would be obtained if the cylinder could be compressed into the longitudinal section through the Y-axis.

ILLUSTRATIVE EXAMPLE.

As an illustration of the last theorem consider the illustrative example of p. 157.

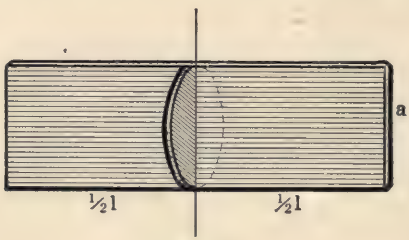


FIG. 91.

Applying the theorem we see that the moment of inertia of the cylinder equals the sum of the moments of inertia of the two laminæ of Fig. 91.

Denoting the moment of inertia of the circular lamina by I_1 and that of the rectangular lamina by I_2 we have

$$\begin{aligned} I &= I_1 + I_2 \\ &= \frac{1}{2}\frac{ma^2}{2} + \int_{-\frac{l}{2}}^{\frac{l}{2}} x^2\, dm \\ &= \frac{ma^2}{4} + \sigma a \int_{-\frac{l}{2}}^{\frac{l}{2}} x^2\, dx \\ &= m\left(\frac{a^2}{4} + \frac{l^2}{12}\right), \end{aligned}$$

which is identical with the result obtained by the direct method.

PROBLEMS.

1. Find the moment of inertia of a hollow circular cylinder with respect to, (a) its geometrical axis, (b) an element, (c) a transverse axis through the center of mass.

2. Find the moment of inertia of an elliptical cylinder with respect to, (a) its geometrical axis, (b) a transverse axis through its center of mass and parallel to the major axis of a right section.

3. Find the moment of inertia of a rectangular prism with respect to, (a) its geometrical axis, (b) a transverse axis through the center of mass and perpendicular to one of its faces.

4. In the preceding problem suppose the prism to be hollow.

5. Find the moment of inertia of a prism, the cross section of which is an equilateral triangle, with respect to, (a) its geometrical axis, (b) a transverse axis through its center of mass and perpendicular to one of its faces.

6. In the preceding problem suppose the prism to be hollow.

7. Find the moment of inertia of a hollow sphere with respect to, (a) a diameter, (b) a tangent line.

8. Find the moment of inertia of a spherical shell of negligible thickness with respect to a tangent line.

9. Find the moment of inertia of a right circular cone with respect to, (a) its geometrical axis, (b) a transverse axis through the vertex.

10. In the preceding problem suppose the cone to be a shell of negligible thickness.

11. Find the moment of inertia of a paraboloid of revolution with respect to, (a) its axis, (b) a transverse axis through its vertex. The radius of the base is a and the height is h.

12. Find the moment of inertia of an ellipsoid with respect to, (a) one of its axes, (b) a tangent at one end of one of the axes parallel to the other.

13. In the preceding problem suppose the body to be an ellipsoidal shell of negligible thickness.

125. General Method. — The special methods which have been discussed in the last few pages are desirable but not necessary for finding the moments of inertia of bodies. Instead of selecting special types of elements of mass for each type of bodies and then making use of the various theorems we can use the general expressions for dm which were given on p. 147 and obtain the moment of inertia directly from equation (II).

As an illustration of this general method consider the moment of inertia of a sphere with respect to a diameter. It is evident from the symmetry of the body that the moment of inertia about a diameter equals eight times that of an octant about one of its straight edges.

(a) Let the octant be taken as shown in Fig. 81, and be referred to Cartesian coördinates, then the equations of the bounding surfaces are

$$x^2 + y^2 + z^2 = a^2, \quad x = 0, \ y = 0, \text{ and } z = 0.$$

Hence taking the x-axis as the axis of reference we have

$$\begin{aligned} I &= \int_0^m r^2\, dm \\ &= 8\tau \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} (y^2 + z^2)\, dx\, dy\, dz \\ &= \tfrac{2}{5} ma^2. \end{aligned}$$

(b) Let the octant be referred to spherical coördinates, Fig. 80, then the equations of the bounding surfaces are

$$r = a, \ \theta = \frac{\pi}{2}, \ \phi = 0, \ \phi = \frac{\pi}{2}, \text{ and } \rho = a.$$

Therefore
$$\begin{aligned} I &= \int_0^m r^2\, dm \\ &= \int_0^m (\rho^2 - \rho^2 \sin^2\theta \cos^2\phi)\, dm \\ &= 8\tau \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^a \rho^4 (\sin\theta - \sin^3\theta \cos^2\phi)\, d\theta\, d\phi\, d\rho \\ &= \tfrac{2}{5} ma^2. \end{aligned}$$

126. Routh's Rule. — The following is a useful rule for remembering the moments of inertia of certain types of bodies:

Moment of inertia with respect to any axis of symmetry

$$= \textit{mass} \times \frac{\textit{sum of the squares of the perpendicular semi-axes}}{\textit{3, 4, 5}}.$$

The denominator of the right-hand member is 3, 4, or 5 according as the body is rectangular, elliptical, or ellipsoidal.

The following illustrate Routh's rule.

Rectangular lamina; about axis perpendicular to its plane:

$$I = m\frac{\frac{a^2}{4} + \frac{b^2}{4}}{3} = m\frac{a^2 + b^2}{12}.$$

Circular lamina; about axis perpendicular to its plane:

$$I = m\frac{a^2 + a^2}{4} = \frac{ma^2}{2}.$$

Elliptical lamina; about axis perpendicular to its plane:

$$I = m\frac{a^2 + b^2}{4}.$$

Rectangular parallelopiped; about axis perpendicular to one of its sides:

$$I = m\frac{\frac{a^2}{4} + \frac{b^2}{4}}{3} = \frac{a^2 + b^2}{12}.$$

Circular cylinder; about longitudinal axis:

$$I = m\frac{a^2 + a^2}{4} = \frac{ma^2}{2}.$$

Sphere; about a diameter:

$$I = m\frac{a^2 + a^2}{5} = \frac{2}{5}ma^2.$$

Ellipsoid; about one of its axes:

$$I = m\frac{a^2 + b^2}{5}.$$

CHAPTER VIII.

WORK.

127. Work. — The mechanical result produced by the action of a force in displacing a particle may be considered to be proportional to the interval of time during which the force acts or to the distance through which it moves. In other words, we can take either the time or the displacement as the standard of measure. The effect measured when the time is taken as the standard is different from that which is obtained when the displacement is made the standard. The first effect is called *impulse.* It will be discussed in a later chapter. The second is called *work,* the subject of this chapter.

128. Measure of Work.—When a force moves a body it is said to do work. The amount of work done equals the product of the force by the distance through which the body is displaced *along the line of action of the force.* In this definition the force is considered to be constant. When it is variable the definition holds for infinitesimal displacements, since during the time taken by an infinitesimal displacement the force may be considered as constant. Therefore if the particle P, Fig. 92, is displaced through ds, under the action of the force **F**, the work done is

$$dW = F \cdot ds \cos \alpha,$$

where α is the angle between the directions of the force and the displacement. When the displacement is finite the work done equals the sum of the amounts of work done in

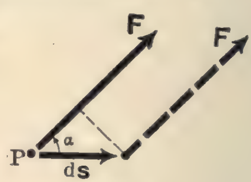


FIG. 92.

successive infinitesimal displacements. Therefore the work done in any displacement is given by the integral

$$W = \int_0^s F \cos \alpha \, ds. \tag{I}$$

When the path of the particle is curved the direction of ds coincides with that of the tangent to the curve. Therefore $F \cos \alpha$ is the tangential component of the force. In other words the tangential component of force does all the work. Hence

$$W = \int^s F_\tau \, ds. \tag{I'}$$

The normal component does no work because the particle is not displaced along it.

Special Cases. Case I.—When the force is constant, both in direction and in magnitude, it can be taken out of the integrand. Therefore

$$W = F \int_0^s \cos \alpha \, ds.$$

The last integral equals the projection of the path upon the direction of the force. Therefore the product of the force by the projection of the path upon the line of action of the force equals the work done.

Case II.—When the force is constant and the path is straight then the angle between the force and the displacement is constant. Therefore

$$W = F \cos \alpha \int_0^s ds$$
$$= Fs \cos \alpha.$$

Case III.—When the force is not only constant but is also parallel to the path, then $\alpha = 0$. Therefore

$$W = Fs.$$

Case IV.—When the force is at right angles to the displacement $\alpha = \frac{\pi}{2}$, and $\cos \alpha = 0$. Hence $W = 0$. Therefore the force does no work unless it has a component along the

path. In this case the motion of the particle is not due to the force in question.

129. Work Done Against the Gravitational Force. — These special cases may be illustrated by considering the work done in raising a body from a lower to a higher level against the gravitational attraction of the earth. Consider the work done in taking a particle from A to B, along each of the three paths shown in Fig. 93.

(a) Suppose the particle to be taken from A to C and then to B; the work done in taking it from A to C comes under Case IV. The direction of motion is at right angles to that of the gravitational force, therefore no work is done against it. The work done in taking the particle from C to B comes under Case III; the force and the motion are in the same direction. Therefore the work done is

$$W = mgh,$$

where h is the vertical height of B above A.

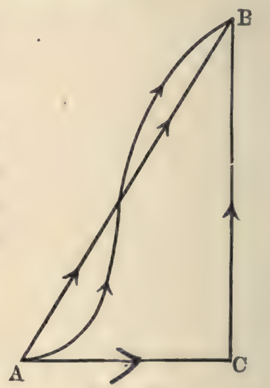


Fig. 93.

(b) Suppose the particle to be taken along the straight line AB. This comes under Case II. The angle between the force and the direction of motion is constant. Therefore

$$W = mgl \cos \alpha,$$

where l is the length of the line AB. But since $l \cos \alpha = h$, the work done is the same as in (a), that is, mgh.

(c) Suppose the particle to be taken along the curve AB. This comes under Case I. Then

$$\begin{aligned} W &= mg \int_0^s \cos \alpha \, ds \\ &= mg \int_0^h dh \qquad [\text{since } ds \cos \alpha = dh] \\ &= mgh. \end{aligned}$$

Therefore *the work done against the gravitational force in taking a body from one position to another depends only upon the vertical height through which the body is raised and not upon the path.*

130. Dimensions and Units of Work. — Work is a scalar magnitude and has for its dimensions $[ML^2T^{-2}]$. The C.G.S. unit of work is the *erg*. It equals the work done by a force of one dyne in displacing a particle through a distance of one centimeter, measured along its line of action. It is symbolized by $\frac{\text{gm.cm.}^2}{\text{sec.}^2}$. The erg is a small unit, therefore a larger unit called the *joule* is also used.

$$1 \text{ joule} = 10^7 \text{ ergs.}$$

The British unit of work is the *foot-pound* (ft.-lb.). It is the work done against the gravitational attraction of the earth in lifting one pound through a vertical distance of one foot. Since the work done in lifting bodies is mgh, we can express the foot-pound in terms of the fundamental units, thus

$$\begin{aligned} 1 \text{ ft. lb.} &= 1 \text{ pd.} \times 32.2 \frac{\text{ft.}}{\text{sec.}^2} \times 1 \text{ ft.} \\ &= 32.2 \frac{\text{pd. ft.}^2}{\text{sec.}^2}, \end{aligned}$$

where pd. represents the pound-mass.

131. Work Done by Components of Force.—The work done by a force $\mathbf{F}$ in giving a particle a displacement $\mathbf{ds}$ is $F \cos \theta \, ds$, where θ is the angle between $\mathbf{F}$ and $\mathbf{ds}$. Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be the rectangular components of $\mathbf{F}$, then the direction of $\mathbf{F}$ is defined by its direction cosines $\frac{X}{F}$, $\frac{Y}{F}$, and $\frac{Z}{F}$. Therefore if l, m, and n denote the direction cosines of $\mathbf{ds}$, we get

$$\cos \theta = l \frac{X}{F} + m \frac{Y}{F} + n \frac{Z}{F}*$$

* See Appendix AIV.

and
$$F \cos \theta \, ds = (lX + mY + nZ)\, ds$$
$$\qquad = X\, dx + Y\, dy + Z\, dr,$$

where **dx**, **dy**, and **dz** are the components of **ds** along the axes. Thus the total work done in a finite displacement is given by

$$W = \int_0^s F \cos \theta \, ds$$
$$= \int_0^x X\, dx + \int_0^y Y\, dy + \int_0^z Z\, dz. \qquad \text{(II)}$$

Equation (II) states that the work done by a force equals the sum of the amounts of work done by its components.

PROBLEMS.

1. Find the number of foot-pounds in one Joule.

2. Find the number of ergs in one foot-pound.

3. Find the work done in dragging a weight w up an inclined plane of length l, height h, and coefficient of friction μ.

4. A body of 100 kg. mass is dragged up, then down, an inclined plane. Compare the work done in the two cases if the length of the plane is 15 m., the height 5 m., and the coefficient of friction 0.5.

5. What is the work done in winding a uniform chain which hangs from a horizontal cylinder? The chain is 25 m. long, and has a mass of 125 kg.

6. A body has to be dragged from a point at the base of a conical hill to a point diametrically opposite. Show that, if the angle which the sides of the hill make with the horizon equals the angle of friction, the work done in dragging the body over the hill is less than in dragging it around the base.

7. A steam hammer falls vertically from a height of 3 feet under the action of its own weight and of a force of 2000 pounds due to steam pressure. At the end of its fall it makes a dent of 1 inch depth in an iron plate. Find the total amount of work done in making the dent. The hammer weighs 1000 pounds.

8. In the preceding problem find the average resisting force.

9. A locomotive which is capable of exerting a draw-bar pull of 1.5 tons is coupled to a train of six cars. The locomotive and the tender weigh 50 tons. The cars weigh 15 tons each. Find the time it takes the locomotive to impart to the train a velocity of 60 miles per hour and the work done under the following conditions.

(a) Horizontal tracks and no resistance.
(b) Horizontal tracks and a resistance of 12 pounds per ton.
(c) Down a grade of 1 in 200 with no resistance.
(d) Same as in (c) but with a resistance of 12 pounds per ton.
(e) Up a grade of 1 in 200 with no resistance.
(f) Same as in (b) but with a resistance of 12 pounds per ton.

10. A mass of 5 pds. is at the bottom of a vertical shaft which reaches the center of the earth. How much work will have to be done in order to bring it to the surface? The weight of a body varies, within the earth, directly as its distance from the center. Take 4000 miles to be the depth of the shaft.

11. Express the result of the last problem in joules.

132. Work Done by a Torque. — Suppose the rigid body A, Fig. 94, to be given an angular displacement $d\theta$ about a fixed axis through the point O, perpendicular to the plane of the paper. The displacement may be considered to be] due to a single force which forms a couple with the reaction of the axis, or it may be considered to be due to small forces acting upon every element of mass of the body.

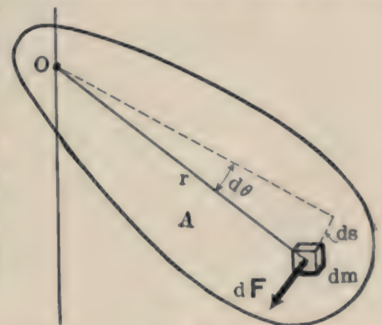


Fig. 94.

Taking the latter view, let **dF** be the resultant force* acting upon the element of mass dm. Then since dm can move only at right angles to the line r, which joins it to the axis, **dF** must be perpendicular to r. When the body is given an angular displacement $d\theta$, dm is displaced through ds, therefore the work done by **dF** is

$$\begin{aligned} d^2W &= dF \cdot ds \\ &= dF \cdot r\, d\theta \\ &= dG\, d\theta, \end{aligned}$$

* **dF** is the resultant of the external forces which act directly on dm, and of the internal forces which are due to the connection of dm with the rest of the body.

where dG is the moment of dF about the axis. Thus the total work done by all the forces acting upon all the elements in producing the angular displacement $d\theta$ is

$$\begin{aligned} dW &= \int_0^G dG\, d\theta \\ &= d\theta \int_0^G dG \qquad \text{($d\theta$ is the same for every element of mass.)} \\ &= G\, d\theta, \end{aligned}$$

where G is the sum of the moments about the axis of all the forces acting upon the elements of the body, i.e., the resultant torque about the axis. The work done in giving the body a finite angular displacement is, therefore,

$$\begin{aligned} W &= \int_0^\theta G\, d\theta \qquad\qquad \text{(III)} \\ &= G\theta \qquad \text{[when G is constant].} \end{aligned}$$

Therefore *work done by a constant torque in producing an angular displacement equals the product of the torque by the angle.*

PROBLEMS.

1. A weight of 10 tons is to be raised by a jackscrew. The pitch of the screw is $\frac{1}{2}$ inch and the length of the bar which is used to turn the nut on the screw is 2 feet long. Supposing the work done by the torque to be expended entirely against gravitational forces, find the force which must be applied at the end of the bar.

2. A ball, which is suspended by a string of negligible mass, is pulled aside until the string makes an angle θ with a vertical line. Show that the work done is the same whether it is supposed to have been done in raising the ball against the action of gravitational forces, or in rotating the ball and the string, as a whole, about a horizontal axis through the point of suspension, against the action of the torque.

3. In the preceding problem take the following data and calculate, by both methods, the amount of work done. Weight of ball = 12 ounces, length of string = 3 feet, and $\theta = 60°$.

4. The torque which has to be applied to the ends of a rod varies directly with the angle through which it is twisted; derive an expression for the work done in turning one end of the rod with respect to the other end through an angle θ.

5. In the preceding problem suppose one end of the rod to be fixed, while the other end is firmly attached to the middle of another rod perpendicular to it. A torqne of 10 pounds-foot is necessary in order to keep the second rod in a position turned through 15° about the axis of the first rod. How much work must be done in order to produce an angular deflection of 45° ?

6. If in problem 5 the torque is due to a couple the forces of which are applied at points 4 inches from the axis of rotation, find the forces applied and show that the work done by the forces equals the work done by the torque.

7. Making the following assumption with regard to the normal pressure at the bearings, obtain an expression for the work done in giving a flywheel an angular displacement.

(*Hint. — For this and the following problems consult* §§ *57 and 58.*)

(a) Normal pressure is constant.

(b) Vertical component of the total reaction is constant.

(c) Normal pressure is a sine function of the angular position; the latter being measured from the horizontal plane through the axis of the shaft.

(d) Normal pressure varies as the square of the sine of the angular position.

8. Find the work done in giving a flywheel a complete rotation. The following data are given. The flywheel weighs 5 tons, the diameter of the shaft is 10 inches, the coefficient of friction in the journal bearings is 0.05, and the normal pressure on the bearings satisfies one of the following conditions:

(a) Normal pressure is constant.

(b) Vertical component of the total reaction is constant.

(c) Normal pressure varies as the sine of the angular position, measured from the horizontal plane through the axis of the shaft.

(d) Normal pressure varies as the square of the angular position.

9. Derive an expression for the work done in giving an angular displacement to a load which is supported by a flat-end pivot.

10. The rotating parts of a water turbine which weigh 50 tons are supported by a flat-end pivot. The diameter of the shaft is 10 inches and the coefficient of friction is 0.03. Find the work lost per revolution.

11. Supposing the normal pressure to be constant derive an expression for the work done in giving a loaded spherical pivot an angular displacement about its axis.

12. Supposing the normal pressure to be constant derive an expression for the work done in giving a loaded conical pivot an angular displacement.

13. A vertical shaft carries a load of 10 tons. Find the work lost per revolution if the shaft is 8 inches in diameter and has a flat-end bearing; the coefficient of friction being 0.01.

14. Derive an expression for the work done in giving a collar-bearing pivot an angular displacement.

15. A vertical shaft carries a load of 5 tons. Find the work lost per revolution if the shaft is supported by a collar-bearing pivot which has an inner diameter of 6 inches and an outer diameter of 8 inches. The coefficient of friction is 0.1.

HOOKE'S LAW.

133. Stress. — When a body is acted upon by external forces which tend to change its shape and thus give rise to forces between its contiguous elements, the body is said to be under *stress*. The measure of *stress is force per unit area:*

$$S = \frac{F}{A}, \tag{IV}$$

where S denotes the stress, F the force, and A the area on which the latter acts.

134. Pressure, Tension, and Shear. — Stresses which occur in bodies are often of a complex nature, but they may be resolved into three component stresses of simple character. These are called *pressure, tension,* and *shear*. Pressure tends to compress, tension to extend, and shear to distort bodies. Shearing stress is the result of a compressive stress combined with a tensile stress at right angles. A special case of shear, which comes into play within a shaft when the latter is twisted, is called *torsion.*

135. Strain. — Strain is the deformation produced by stress. The measure of *strain is the percentage deformation.* For instance, if the deformation consists of a change in length the strain equals the ratio of the increase in length, to the original length:

$$s = \frac{l}{L}, \tag{V}$$

where s denotes the stress, l the increase in length, and L the original length.

136. Hooke's Law. — The relation which connects a stress with the strain which it produces is known as Hooke's law. It states that *stress is proportional to strain:*

$$S = \lambda s, \tag{VI}$$

where λ is the constant of proportionality, and is called the *modulus of elasticity.*

137. Elastic Limit. — Hooke's law holds true so long as stress is small enough to leave no appreciable permanent deformation. In other words, Hooke's law holds true strictly only while the body under consideration behaves like a perfectly elastic body under the action of the given stresses. All bodies are more or less imperfectly elastic; that is, stresses always leave bodies with permanent strains. Therefore at the best Hooke's law is approximately true when applied to material bodies. The approximation, however, is close enough for practical purposes so long as the permanent deformation is negligible compared with the total deformation produced by the stress. If a considerable portion of the deformation becomes permanent the body under stress is said to have reached its *elastic limit*, when Hooke's law does not give a close enough approximation and consequently cannot be used.

138. Young's Modulus. — The modulus of elasticity of a body which is being stretched is called *Young's modulus.* Let the body be an elastic string, a wire, or a rod, and let A be the area of its cross-section, L its natural or unstretched length, and l the increase in length due to stretching. Then we have

$$S = \frac{F}{A} \text{ and } s = \frac{l}{L}.$$

Therefore

$$\frac{F}{A} = \lambda \frac{l}{L},$$

or

$$\lambda = \frac{F}{A} \cdot \frac{L}{l}.$$

Thus Young's modulus of a substance equals, numerically, the force necessary to stretch a uniform rod of unit cross-section, which is made of the given substance, to double its length. During the process of stretching Hooke's law is, of course, supposed to hold.

139. Work Done in Stretching an Elastic String. — Let L denote the natural length of the string and x its length at any instant of the process of stretching. Then the work done in increasing the length by dx is

$$\begin{aligned} dW &= T\,dx \\ &= AS\,dx, \end{aligned}$$

where T is the tensile force, S the tension, and A the area of the cross-section of the string. But by Hooke's law,

$$S = \lambda s.$$

In this case $s = \dfrac{x-L}{L}$. Therefore

$$dW = A \cdot \lambda \frac{x-L}{L} \cdot dx$$

and

$$\begin{aligned} W &= \frac{\lambda'}{L}\int_L^{L+l}(x-L)\,dx \\ &= \frac{\lambda'}{2L}l^2, \end{aligned}$$

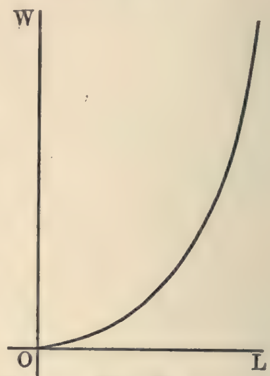


FIG. 95.

where $\lambda' = A\lambda$, and l is the total increase in length. Thus the work done varies as the square of the increase in length. Plotting l as abscissa and W as ordinate we obtain a parabola, Fig. 95.

140. Work Done in Compressing Fluids. — Let C, Fig. 96, be a cylinder which contains a compressible fluid and which is provided with a piston. When the piston is displaced toward the left work is done against the force with which the fluid presses upon the piston. If dx denotes the dis-

placement and F the force on the piston then the work done is

$$\begin{aligned} dW &= -F\,dx \\ &= -pA\,dx \\ &= -p\,dv, \end{aligned}$$

where p is the pressure in the fluid, A the area of the piston, and dv the change in the volume of the fluid. Therefore the total work done in compressing the fluid from a volume v_1 to a volume v_2 is

$$W = -\int_{v_1}^{v_2} p\,dv. \tag{1}$$

When the law connecting p and v is given the work done in compressing or expanding a fluid can be found by carrying out the integration indicated in equation (1). During expansion, however, the displacement has the same direction as the force which causes the expansion; therefore the sign before the integral is positive.

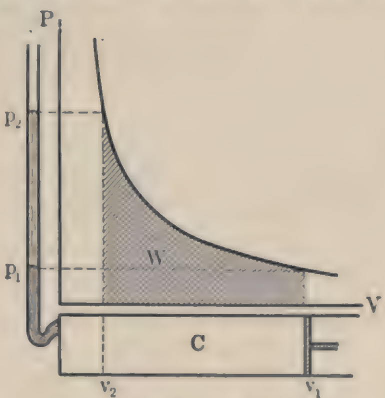


FIG. 96.

141. Representation of the Work Done in the *PV*-Diagram.— When the volume of the expanding fluid is plotted as abscissa and the pressure as ordinate, a curve is obtained, which represents, graphically, the law connecting p and v. Such a representation is called a *PV-diagram*. It is evident from equation (1) that the area bounded by the curve, the v-axis, and the two vertical lines whose equations are $v = v_1$ and $v = v_2$, represents the work done in compressing the fluid from v_1 to v_2.

142. Isothermal Compression of a Gas. — If a gas is compressed without changing its temperature the compression

is called *isothermal*, in which case the relation between p and v is given by Boyle's law, i.e.,

$$pv = k. \tag{2}$$

Substituting in equation (1) the value of p given by equation (2) we obtain

$$\begin{aligned} W &= -k\int_{v_1}^{v_2}\frac{dv}{v} \\ &= k\log\frac{v_1}{v_2}. \end{aligned} \tag{3}$$

143. Adiabatic Compression of a Gas.—If no exchange of heat is allowed between the gas and other bodies while the former is being compressed the compression is called *adiabatic.* The law which connects p and v in an adiabatic compression or expansion of a gas may be expressed by the relation

$$pv^\gamma = k, \tag{4}$$

where γ and k are constants for a given gas. Substituting in equation (1) the value of p, which is given by equation (4), we obtain

$$\begin{aligned} W &= -k\int_{v_1}^{v_2}\frac{dv}{v^\gamma} \\ &= \frac{k}{\gamma-1}\left(v_2^{1-\gamma} - v_1^{1-\gamma}\right) \\ &= \frac{p_2v_2 - p_1v_1}{\gamma-1}. \end{aligned} \tag{5}$$

144. Modulus of Elasticity of a Gas.—Let $-dv$ denote the change in volume due to an increase in the pressure of a gas by an amount dp. Then the stress is dp and the strain $\dfrac{-dv}{v}$. Therefore by Hooke's law

$$dp = \lambda\frac{-dv}{v}$$

or

$$\lambda = -v\frac{dp}{dv}. \tag{6}$$

The modulus of elasticity λ is not a definite constant for a given gas, because the value of $\frac{dp}{dv}$ depends upon the temperature and the amount of heat of the gas. Therefore the state of a gas for which $\frac{dp}{dv}$ is calculated should be stated in order that the value of λ may have any meaning at all. There are two states for which λ is calculated, namely, the isothermal and the adiabatic states.

145. Isothermal Elasticity. — When the compression is isothermal

$$pv = k$$

and

$$\frac{dp}{dv} = -\frac{k}{v^2}.$$

$$\therefore \quad \lambda = \frac{k}{v} = p. \tag{7}$$

Therefore the isothermal elasticity of a gas numerically equals the pressure.

146. Adiabatic Elasticity. — When the gas is compressed adiabatically

$$pv^{\gamma} = k$$

and

$$\frac{dp}{dv} = -k\gamma v^{-\gamma-1}$$

$$= -\gamma p v^{-1}.$$

$$\therefore \quad \lambda = \gamma p. \tag{8}$$

147. Torsional Rigidity of a Shaft. — Suppose the upper end of the cylinder of Fig. 97 to be rotated about the axis of the cylinder through an angle θ, while the lower end is fixed, and consider the stresses and the strains in the cylinder. It is evident that the strain is nil at the axis and increases uniformly with the distance from the axis. Further the strain is nil at the lower base and increases uniformly with the distance from it. Since Hooke's law holds these statements are true with regard to the stress in the cylinder.

Let dF denote the force acting on the area, on the upper base, of a ring of radius r and width dr, then the stress equals $\frac{dF}{2\,\pi r \cdot dr}$. But if θ is the angle of twist at the upper base and l the length of the cylinder, then the strain equals $\frac{r\theta}{l}$. Therefore by Hooke's law

$$\frac{dF}{2\,\pi r\,dr} = \lambda\frac{r\theta}{l}.$$

In this case λ is called *modulus of shearing elasticity* or, simply, *shear modulus*. Solving the preceding equation for dF we get

$$dF = \frac{2\,\pi\lambda}{l}\,\theta r^2\,dr.$$

Therefore the torque acting upon the area of the ring is

$$\begin{aligned} dG &= r \cdot dF \\ &= \frac{2\,\pi\lambda}{l}\,\theta r^3\,dr. \\ \therefore \quad G &= \frac{2\,\pi\lambda}{l}\,\theta\int_0^a r^3\,dr \\ &= \lambda\frac{\pi a^4}{2\,l}\,\theta, \end{aligned} \tag{9}$$

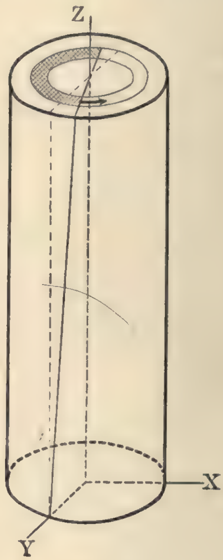


FIG. 97.

where G is the total torque applied at the upper end and a the radius of the cylinder. Thus the torque necessary to produce a given angle of twist varies directly as the fourth power of the radius and inversely as the length. On the other hand for a given shaft the torque varies directly with the angle of twist.

The *torsional rigidity* of the shaft is defined as the torque necessary to produce a unit angular twist; therefore

$$\begin{aligned} R &= \frac{G}{\theta} \\ &= \lambda\frac{\pi a^4}{2\,l}. \end{aligned} \tag{10}$$

It will be observed that the torsional rigidity of a solid shaft varies directly as the fourth power of the radius and inversely as the length.

148. Work Done in Twisting a Rod. — Work done by a torque is obtained by substituting the expression for the torque in the work equation. Thus

$$W = \int_0^\theta G\,d\theta.$$

$$= \frac{\lambda\pi a^4}{2\,l}\int_0^\theta \theta\,d\theta \qquad \text{[by eq. (9).]}$$

$$= \tfrac{1}{2}\,k\theta^2, \qquad (11)$$

where $$k = \lambda\frac{\pi a^4}{2\,l}.$$

PROBLEMS.

1. What are the dimensions of stress, strain, and modulus of elasticity?

2. A steel rod of $\frac{1}{2}$-inch radius is found to stretch 0.004 inch in 10 inches of its length when a load of 10,000 pounds is gradually applied. Find the Young's modulus of the rod.

3. The Young's modulus of a brass wire is $10.8 \times 10^{11}\,\dfrac{\text{dynes}}{\text{cm.}^2}$. Find the load (in pounds) necessary in order to produce an elongation of 0.5 mm. in 1 meter. The diameter of the wire is 1 millimeter.

4. The modulus of shearing elasticity of a steel shaft is 11×10^6 pounds per square inch. What force acting at the end of a lever 30 inches long will twist asunder the shaft if it is 0.5 inch in diameter?

5. A brass rod, 4 feet long and 1.5 inches in diameter, is twisted through an angle of 9° by a force of 1500 pounds acting 6 inches from the axis of the rod. If on removal of the stress the bar recovers its original position, calculate the modulus of shearing elasticity of the rod.

6. Taking the data of the preceding problem find the force necessary to give an angle of twist of 2° to a rod 15 inches long, 0.5 inch in diameter.

7. An elastic string of natural length l is stretched to twice its length when it supports a weight W. The ends of the string are connected to two points at the same level and a distance d apart, while the weight W is attached to the middle of the string. Find the position of equilibrium of the weight.

8. A spider hangs from the ceiling by a thread which is stretched by the weight of the spider to twice its natural length. Show that the work

done by the spider in climbing to the ceiling equals $\frac{3}{4}\, mgh$, where m is the mass of the spider and h its distance from the ceiling.

9. The outer end of a flat spiral spring is fixed, while the inner end is attached to the center of a bar 20 cm. long, in such a way that the bar is parallel to the plane of the spring. Two forces of 500 dynes each applied at the ends of the bar, at right angles to the bar and parallel to the plane of the spring, can keep the bar turned through an angle of $\frac{\pi}{2}$ radians. What torque must be applied in order to keep the bar in position after giving it three turns?

10. In the preceding problem find the work done in giving the bar three turns. What portion of the total work is done in the last turn?

11. Prove that the following is the expression for the torsional rigidity of a hollow shaft:

$$R = \lambda \frac{\pi\,(a^4 - b^4)}{2\,l},$$

where b is the inner radius of the shaft, while the other letters represent the same magnitudes as in § 147.

12. Derive expressions for the saving of material and loss of rigidity due to making a shaft of a given external diameter hollow.

13. Find the value of the quotient of the inner to the outer radius which will make the quotient of the saving of material to the loss of rigidity a maximum.

14. The weight and the length of a shaft are fixed; find the ratio of the inner to the outer diameter which will make the rigidity of the shaft a maximum.

15. The torsional moment which a shaft has to withstand and the length are fixed; find the ratio of the inner to the outer diameter which will make the weight of the shaft a minimum.

VIRTUAL WORK.

149. Principle of Virtual Work. — The concept of work enables us to formulate a principle, called *the principle of virtual work,* which can be applied to equilibrium problems to great advantage.

In order to derive this principle consider a particle which is in equilibrium. Evidently the resultant force acting upon the particle is nil and remains nil so long as the particle is in the equilibrium position. But when the particle is given a

small displacement, the resultant force assumes a value different from zero. If the displacement is small enough, so that the departure from equilibrium position and consequently the resultant force remains small, the displacement is called a *virtual displacement* and the work done by the resultant force *virtual work*. We will call *virtual force* the small resultant force, which is called into play by the virtual displacement.

Let F_1, F_2, etc., be the forces under the action of which the particle is in equilibrium. When the particle is given a virtual displacement ds, these forces are changed, in general, in magnitude and direction so that a virtual force dF acts upon the particle during the displacement. Then the virtual work is

$$dF \cdot ds = F_1 \cdot ds_1 + F_2 \cdot ds_2 + \cdots, \qquad \text{(VII)}$$

where ds_1, ds_2, etc., are the displacements of the particle along the forces F_1, F_2, etc., due to the virtual displacement ds. But since the left-hand member of the last equation is an infinitesimal of the second order while the terms of the right-hand member are infinitesimals of the first order we can neglect the left-hand member and write

$$F_1\, ds_1 + F_2\, ds_2 + \cdots + = 0. \qquad \text{(VIII)}$$

Equation (VIII) states: *when a particle which is in equilibrium is given a virtual displacement the total amount of work done by the forces acting upon the particle vanishes.* This is the principle of virtual work.

The principle of virtual work is applicable not only to particles, but also to any system which is in equilibrium. If the system is acted upon by torques as well as forces, then the sum of the work done by the virtual torques and the virtual forces vanishes:

$$F_1\, ds_1 + F_2\, ds_2 + \cdots + G_1\, d\theta_1 + G_2\, d\theta_2 + \cdots = 0. \qquad \text{(IX)}$$

ILLUSTRATIVE EXAMPLES.

1. Supposing the weights in Fig. 98 to be in equilibrium and the contacts to be smooth, find the relation between the two weights.

If W_1 is given a virtual displacement towards the left along the inclined plane, then the virtual work is

$$-T\,ds + W_1 \cdot ds \sin\alpha + N \cdot 0 = 0,$$

or $\qquad T = W_1 \sin\alpha.$

But $\qquad T = W_2.$

Therefore $\qquad W_2 = W_1 \sin\alpha.$

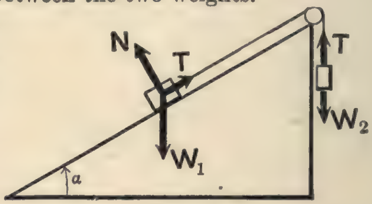


FIG. 98.

2. Two uniform rods of equal weight W and equal length a are jointed at one end and placed, as shown in Fig. 99, in a vertical plane on a smooth horizontal table. A string of length l joins the middle points of the rods. Find the tension of the string.

The following forces act upon each rod — the weight of the rod, the pull of the string, the reaction at the joint, and the reaction of the table. Suppose a slight displacement to be given to the system by pressing downward at the joint. The work done by the force which produced the displacement equals the sum of the work done by the other forces which act upon the rods during the displacement. But since both the force applied and the displacement produced are very small their product is negligible. Therefore the sum of the work done by all the other forces is zero.

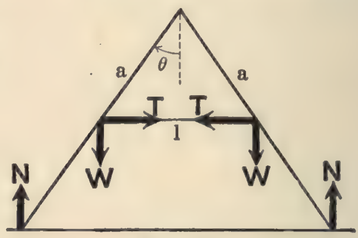


FIG. 99.

The reactions at the ends of the rods do not contribute to the virtual work because each of the reactions is perpendicular to the corresponding surface of contact along which the displacement takes place. Therefore the weights and the tensile force of the string contribute all the virtual work. If dl and dh denote, respectively, the increase in length of the string and the distance through which the centers of mass of the rods are lowered during the virtual displacement the virtual work takes the form

$$2\left(T \cdot \frac{dl}{2} + W\,dh\right) = 0.$$

But from the figure $l = a \sin\theta$, and $h = a\cos\theta$. Therefore $dl = a\cos\theta\, d\theta$ and $dh = -\,a \sin\theta\, d\theta$. Making these substitutions and simplifying we obtain

$$T = 2\,W \tan\theta.$$

3. Find the mechanical advantage of the jack-screw.

Let p be the pitch of the screw, l the length of the lever arm, F the force applied and P the force derived. Then since at any instant the system is supposed to be in equilibrium the virtual work, due to a small displacement, must vanish. Let $d\theta$ denote a small angular displacement and dh the corresponding rise of the screw. Then if G denotes the torque applied the virtual work takes the form

$$G\,d\theta - P\,dh = 0.$$

But $G = F \cdot l$ and $dh = \dfrac{d\theta}{2\pi}p$. Therefore

$$Fl\,d\theta = \frac{Pp\,d\theta}{2\pi}.$$

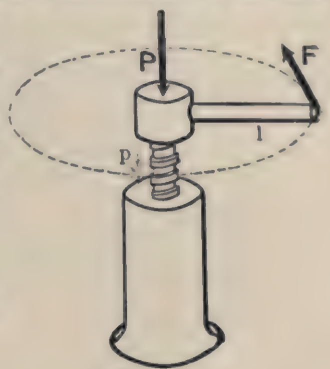


FIG. 100.

Hence the mechanical advantage, which is the quotient of the force derived to the force applied, is

$$\frac{P}{F} = \frac{2\pi l}{p}.$$

PROBLEMS.

1. By the application of the principle of virtual work derive the expression for the mechanical advantage of

(a) the lever;
(b) the wheel and axle;
(c) the hydraulic press;
(d) the pulley (a) of problem 13 on page 21;
(e) the pulley (b) of problem 13 on page 21;
(f) the pulley (c) of problem 13 on page 21;
(g) the pulley (d) of problem 13 on page 21.

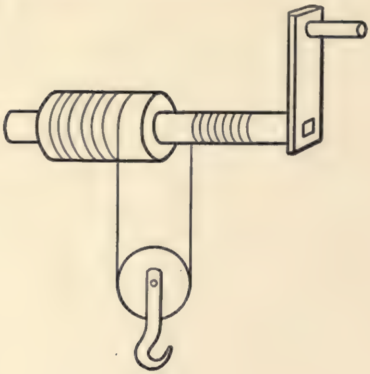
2. Apply the principle of virtual work to

(a) illustrative problem 1 on page 17;
(b) problem 4 on page 20;
(c) problem 6 on page 46;
(d) problem 16 on page 47.

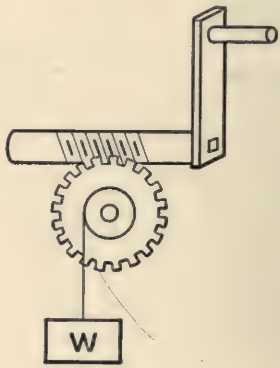
3. Four rods of equal weight W are freely jointed so as to form a square. The system is suspended vertically from one of the joints. A string of negligible weight connects two of the joints so as to keep the square shape of the system. Find the tension of the string.

4. An elastic band of weight W and natural length l is slipped over a smooth circular cone the axis of which is vertical. The force necessary to stretch the string to double its natural length equals λ. Find the position of equilibrium and the tension of the string.

5. Find the mechanical advantage of the following machines.



(a)



(b)

CHAPTER IX.

ENERGY.

150. Results of Work. — Consider the work done by the engine of a train in pulling it upgrade. The work done may be divided into three parts:

(a) Work done against frictional forces.
(b) Work done against gravitational forces.
(c) Work done against the kinetic reaction.

The result of work done against frictional forces is heat. The amount of heat generated is proportional to the amount of work done. The heat may be utilized, at least theoretically, to do work. Thus a part, if not all, of the original work may be recovered.

The apparent result of the work done against the gravitational forces is the elevation of the train to a higher level. The work done may be recovered by letting the train come down to its former level and thereby do work. Therefore the work done against gravitational forces may be considered to be stored up.

The apparent result of the work done against the kinetic reaction in accelerating the train is an increase in the velocity of the train. The work done may be recovered by letting the train overcome a force, which tends to reduce the velocity of the train to its original value. Therefore in this case also the work done may be said to have been stored up. In fact in all three cases the work done is stored up. In the first case, however, work is not available as readily as in the other two cases. In order to convert

heat into work special means, such as heat engines, etc., have to be used, which do not belong to the domain of ordinary mechanics; therefore work done against frictional forces is considered as lost. On the other hand work which is done against gravitational forces or against kinetic reactions is directly available for mechanical work.

151. Energy. Potential, Kinetic, and Heat Energy.—Energy may be defined as work which is stored up. Work stored up in overcoming kinetic reactions is called *kinetic energy*. Work stored up while overcoming nonfrictional forces, such as gravitational forces, is called *potential energy*. Work done while overcoming frictional forces is called *heat energy*.

152. Transformation of Energy. — Potential, kinetic, and heat energy are different (at least apparently*) forms of the same physical entity, i.e., energy. Energy may be changed from any one of these forms into any other form. Whenever such a change takes place energy is said to be *transformed*. Transformation of energy is always accompanied by work. In fact the process of doing work is that of transformation of energy. The amount of energy transformed equals the amount of work done.

The units and dimensions of energy are the same as those of work.

KINETIC ENERGY.

153. Kinetic Energy of a Particle. — By definition kinetic energy equals the work done against the kinetic reaction in giving the particle its velocity. Since there is no motion along the normal to the path of the particle no work is done against the normal component of the kinetic reaction. Therefore we need only consider the work done against the tangential component.

* Recent developments in physical sciences tend to show that differences between different forms of energy are only apparent and that all forms of energy are, in the last analysis, kinetic.

Denoting the kinetic energy by T and putting the definition into analytical language we obtain

$$T = -\int_0^s \left(-m\frac{dv}{dt}\right) ds^*$$
$$= m\int_0^v \frac{ds}{dt}\, dv$$
$$= \tfrac{1}{2}\, mv^2. \qquad \text{(I)}$$

Therefore the *kinetic energy of a particle equals one-half the product of the mass by the square of its velocity.* Since both m and v^2 are positive, kinetic energy must be a positive magnitude. The kinetic energy of a system of particles, therefore, equals the arithmetic sums of the kinetic energies of the individual particles. Thus

$$T = \tfrac{1}{2}\,\Sigma mv^2. \qquad \text{(II)}$$

When all the particles of the system have the same velocity

$$T = \tfrac{1}{2}\, Mv^2,$$

where M is the total mass of the system.

154. Work Done in Increasing the Velocity of a Particle. — If the velocity of a particle is increased from v_0 to v then the work done against the kinetic reaction equals the increase in the kinetic energy of the particle. This will be seen from the following analysis:

$$W = -\int_0^s \left(-m\frac{dv}{dt}\right) ds$$
$$= m\int_{v_0}^v v\, dv$$
$$\left.\begin{aligned} &= \tfrac{1}{2}\, mv^2 - \tfrac{1}{2}\, mv_0^2 \\ &= T - T_0. \end{aligned}\right\} \qquad \text{(III)}$$

* The first negative sign indicates the fact that T is the work done against and not by the kinetic reactions.

The second negative sign belongs to the kinetic reaction as it was explained in Chapter VI.

PROBLEMS.

1. Show that the dimensions of work and kinetic energy are the same.

2. A body of 50 gm. mass starts from the top of an inclined plane 10 m. long, and arrives at the bottom with a velocity of 300 $\frac{\text{cm.}}{\text{sec.}}$. Find the average frictional force. The angle of elevation of the plane is 30°.

3. A body of 100 gm. mass, which is projected up an inclined plane, arrives at the top of the plane with a velocity of 150 $\frac{\text{cm.}}{\text{sec.}}$. Find the velocity of projection, supposing the frictional force to be constant and equal to 10,000 dynes. The length of the plane is 5 inches, and the angle of elevation is 30°.

4. A bullet enters a plank with a velocity of 1500 $\frac{\text{ft.}}{\text{sec.}}$, and leaves it with a velocity of 1350 $\frac{\text{ft.}}{\text{sec.}}$. How many such planks can the bullet penetrate?

5. In the preceding problem find the average resisting force which the planks offer. The bullet weighs ½ ounce.

6. A catapult, which consists of an elastic string 15 cm. long, with its ends tied to the prongs of a forked piece of wood, is used to throw stones. What velocity will it give to a stone of 5 gm. mass when stretched to twice its natural length. The modulus of elasticity of the string is 2 pounds.

7. The kinetic energy acquired by a weight of 750 pounds in falling through a distance of 4 feet is to be absorbed by a helical spring, 5 inches long. Find the modulus of elasticity of the spring so that it will not be compressed more than 1 inch.

8. Having a given size and shape, how will the penetrative power of a bullet depend (a) on its weight, and (b) on its velocity. The resisting force is supposed to be constant.

155. Kinetic Energy of a Rigid Body Rotating About a Fixed Axis. — Suppose the rigid body A, Fig. 101, to rotate about an axis through the point O, at right angles to the plane of the paper. Consider the kinetic energy of an element of mass dm at a distance r from the axis. If $\mathbf{v}$ denotes the velocity of the element and dT its kinetic energy, then

$$\begin{aligned} dT &= \tfrac{1}{2}\, v^2\, dm \\ &= \tfrac{1}{2}\, r^2\omega^2\, dm, \quad [v = r\omega] \end{aligned}$$

where ω is the angular velocity of the body. Therefore the total kinetic energy of the rotating body is

$$T = \tfrac{1}{2}\int_0^m r^2\,dm \cdot \omega^2$$
$$= \tfrac{1}{2} I\omega^2, \qquad \text{(IV)}$$

where I is the moment of inertia of the body about the axis of rotation.

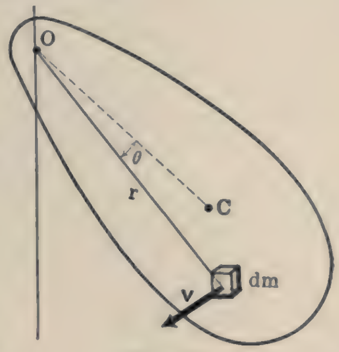


FIG. 101.

Comparing the expression for the kinetic energy of rotation with the expression for the kinetic energy of translation we observe that moment of inertia plays the same role in motion of rotation as mass, the linear inertia, plays in motion of translation.

The expression for the kinetic energy of a rotating body may be put in a little different form by substituting for I its value in terms of the moment of inertia about a parallel axis through the center of mass. Thus

$$\begin{aligned} T &= \tfrac{1}{2} I\omega^2 \\ &= \tfrac{1}{2}(I_c + ma^2)\,\omega^2 \\ &= \tfrac{1}{2} ma^2\omega^2 + \tfrac{1}{2} I_c\omega^2 \\ &= \tfrac{1}{2} mv_c{}^2 + \tfrac{1}{2} I_c\omega^2, \qquad \text{(V)} \end{aligned}$$

where v_c is the velocity of the center of mass. We have thus divided the kinetic energy into two parts — (a) kinetic energy due to the motion of translation of the body as a whole with the velocity of the center of mass, (b) kinetic energy due to the rotation of the body about an axis through the center of mass.

156. Work Done in Increasing the Angular Velocity of a Rigid Body. — It was shown in § 154 that the work done against the kinetic reaction of a particle equals the increase in the kinetic energy of the particle. Therefore the work done against the kinetic reaction of any number of particles is the

sum of the increments in the kinetic energies of the individual particles. Therefore

$$W = \Sigma\left(\tfrac{1}{2}\, mv^2 - \tfrac{1}{2}\, mv_0^{\,2}\right).$$

When the particles form a continuous system, we can replace the particles by elements of mass and the summation sign by the integration sign. Thus

$$\begin{aligned} W &= \int_0^m \left(\tfrac{1}{2}\, v^2\, dm - \tfrac{1}{2}\, v_0^{\,2}\, dm\right) \\ &= \tfrac{1}{2}\int_0^m \left(r^2\omega^2\, dm - r^2\omega_0^{\,2}\, dm\right) \\ &= \tfrac{1}{2}\,\omega^2 \int_0^m r^2\, dm - \tfrac{1}{2}\,\omega_0^{\,2}\int_0^m r^2\, dm \\ &= \tfrac{1}{2}\, I\omega^2 - \tfrac{1}{2}\, I\omega_0^{\,2}, \end{aligned} \tag{VI}$$

where ω_0 and ω are the initial and the final values of the angular velocity of the body. Therefore in this case also work done equals the increase in the kinetic energy.

PROBLEMS.

1. The flywheel of a metal punch is 4 feet in outside diameter and weighs 500 pounds. What must be its initial velocity in order that the punch may exert a force of 25 tons, through a distance of 1 inch, without reducing the speed of the flywheel by more than 25 per cent? Neglect the effect of the shaft, and consider the flywheel to be a disk.

2. The power of a 15-ton car was shut off and the brakes were put on at a time when the car was making 50 miles an hour. On each of the 8 wheels a normal brake-shoe force of 5000 pounds was applied. Find the distance covered by the car before coming to rest. The diameters of the wheels are 30 inches, the tracks are horizontal, and the coefficient of friction for the contact between the shoes and the wheels is 0.2.

3. A 100-ton locomotive making a mile a minute is to be stopped within 500 yards. What brake-shoe force must be applied? The diameters of the wheels are 6 feet. The coefficient of friction is 0.3.

4. Find the amount of heat which would be generated if the rotation of the earth about its axis were stopped. The mean density of the earth $= 5.5\,\dfrac{\text{gm.}}{\text{cm.}^3}$, the radius = 4000 miles; 1 calorie = 4.2 $(10)^7$ ergs.

5. How many cubic miles of ice could be melted by the heat computed in the preceding problem? The latent heat of ice is 80 calories per gram.

6. The winder of a spinning top is a helical spring, which is set in a cylindrical piece 1 inch in diameter. When the winder is hooked to the top and twisted through π radians a force of 1 pound has to be applied to the cylinder tangentially in order to keep it from untwisting itself. After the spring is given a twist of $2\frac{1}{2}$ turns the top is released. Find the kinetic energy the top would acquire if there were no frictional forces.

7. In the preceding problem find the angular velocity of the top supposing it to consist of a circular plate of 2 inch radius, and of $\frac{1}{4}$ pound weight.

8. If the top of the preceding problem turns for 2 minutes before stopping, find the mean torque due to friction and resistance; also find the total number of revolutions made.

9. A top is given a motion of rotation by pulling at a string wound around it. Derive an expression for the energy communicated, (a) when the force applied to the string is constant; (b) when it varies directly with the length of the string which is unwound.

POWER.

157. Power. — Power is the rate at which work is done. When put into the language of calculus this definition becomes

$$P = \frac{dW}{dt}. \qquad \text{(VII)}$$

Power is a scalar quantity and has the dimensions $[ML^2T^{-3}]$. The C.G.S. unit of power is the erg per second. This unit is too small for engineering purposes; therefore two larger units are adopted, which are called the *watt* and the *kilowatt*. The following relations define these units:

$$1 \text{ watt} = \frac{1 \text{ joule}}{1 \text{ sec.}}$$

$$= 10^7 \frac{\text{ergs}}{\text{sec.}}.$$

$$1 \text{ kilowatt} = 10^3 \text{ watts}$$

$$= 10^{10} \frac{\text{ergs}}{\text{sec.}}.$$

The British unit of power is the *horse power*, defined by the following equation:

$$1 \text{ H.P.} = 33{,}000 \frac{\text{ft. lbs.}}{\text{min.}}$$
$$= 550 \frac{\text{ft. lbs.}}{\text{sec.}}.$$

PROBLEMS.

1. Show that 1 horse power equals about 746 watts.

2. The engine of a train, which weighs 150 tons, is of 200 horse power. Find the maximum speed the train can attain on a level track if there is a constant resisting force of 15 pounds per ton.

3. The diameter of the cylinder of a steam engine is 9 inches, and its length 10 inches; the mean effective pressure per square inch is 90 pounds, and the number of revolutions per minute is 100. Find the indicated horse power.

4. Each of the 2 cylinders of a locomotive is 16 inches in diameter, the length of the crank is 9 inches, the diameter of the driving wheels is 6 feet, the velocity of the train is 40 miles per hour, and the mean effective pressure is 75 pounds per square inch. Find the power developed.

5. A train weighing 125 tons moves at the rate of 50 miles an hour, along a horizontal road. Find the power, in kilowatts, transformed by the motors of the electric engine which pulls the train. The resistance is 10 pounds per ton.

6. Find the horse power developed by an engine which moves a train at the rate of 30 miles an hour up an incline of 1 in 300. The train weighs 120 tons and there is a resistance of 15 pounds per ton.

7. A belt traveling at the rate of 45 feet per second transmits 100 horse power. What is the difference in tension of the tight and the slack sides of the belt. The width of the belt is 20 inches.

8. A 150-horse-power steam engine has a piston 18 inches in diameter which makes 100 strokes per minute. Find the mean effective pressure of the steam in the cylinder. The length of the stroke is 24 inches.

9. The average flow over the Niagara Falls is 10,000 cubic meters per second. The average height is 160 feet. Find the power, in kilowatts, which could be generated if all the energy were utilized.

10. A fire engine pumps water with a velocity of 125 $\frac{\text{ft.}}{\text{sec.}}$ through a nozzle 1 inch in diameter. Find the horse power of the engine required to drive the pump, if the efficiency of the pump is 75 per cent and the

nozzle is 15 feet above the surface of the reservoir which supplies the water.

11. Find the power of a machine gun which projects 600 bullets per minute with a muzzle velocity of 500 $\frac{\text{m.}}{\text{sec.}}$ and angular velocity of 600 π radians per second. The bullets are cylinders 0.9 cm. in diameter and 15 gm. mass.

12. A shaft transmits 50 horse power and makes 150 revolutions per minute. Express the torque transmitted in pounds-foot and dynes-cm.

13. An electric motor develops 25 kilowatts at 900 revolutions per minute. Find the torque on the rotating armature due to the field magnets. Neglect friction.

14. Find the power of a clock which has a maximum run of 8 days. The weight which moves the works has a mass of 10 kg. At its highest position the weight is 15 inches above its lowest position.

15. A twin-screw steamer has engines of 20,000 horse power and when working at full power the engines make 75 revolutions per minute. Find the torque transmitted by the shaft of each screw.

16. The pitch of the screw propeller of a ship is 25 feet. The power transformed by the propeller is 15,000 horse power, when the ship makes 20 knots. Assuming that there is a slip of 10 per cent at the propeller screw and that the efficiency is 0.75, find the torque transmitted by the shaft, also the thrust on the bearings.

17. A feed pump delivers water into a boiler at the rate of 20 lbs. an hour. If the pressure in the boiler is 150 lbs. per square inch above the atmospheric pressure, find the effective horse power of the pump.

POTENTIAL ENERGY.

158. Configuration. — The arrangement of the parts of a system is called the *configuration* of the system. The system which consists of this book and the earth, for instance, is in one configuration when the book is on the desk and in another configuration when it is on the floor. During the transfer of the book from the floor to the desk the system passes, continuously, through infinite number of configurations, because the book occupies infinite number of different positions relative to the earth.

159. Conservative Forces. — If the work done in bringing a system from one configuration to another configuration is

independent of the manner in which the change of configuration takes place, the forces acting upon the system are said to be *conservative forces.* Gravitational forces are examples of conservative forces. This is evident from the result of § 129 where it was shown that the work done against gravitational forces in taking a body from one point to another is independent of the path along which the body is carried.

160. Dissipative Forces. — Forces which are not conservative are called *dissipative* or *nonconservative forces.* All frictional and resisting forces are of this type.

161. Potential Energy. — The potential energy of a system in any configuration equals the work done against the conservative forces which act upon the system, in bringing it from a *standard configuration* to the configuration in question. For instance, if the unstretched state of an elastic string is taken to be its standard configuration, then the potential energy of the string at any stretched state equals the work done in producing the extension. The potential energy of this book when on the table equals the work done in raising it from the floor to the table, provided the book is considered to be at the standard configuration when it is on the floor.

The selection of the standard configuration is quite arbitrary and is a matter of convenience only.

It is evident from the definition of potential energy that its value is zero at the standard configuration.

Comparing the definitions of potential energy and of conservative forces we see that the potential energy at any given configuration is independent of the manner in which the system is brought from the standard configuration. This is equivalent to stating that the potential energy of a system depends upon its configuration. But coördinates define the configuration of a system; therefore *potential energy is a function of the coördinates.*

If the sea level is taken as the standard configuration, i.e., the position of zero potential energy, then the potential

energy of a body, due to gravitational forces, is a function of the vertical height of the body above the sea level; in fact it equals mgh, where mg is the weight of the body and h its height above the sea level.

162. Difference of Potential Energy. — The difference between the potential energy of a system in two different configurations equals the work done in taking the system from the configuration of lower potential energy to that of higher potential energy.

Let the point A, Fig. 102, represent the standard configuration and the points B and C represent two other con-

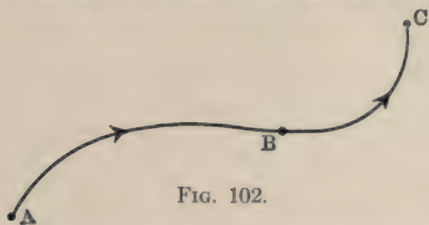


Fig. 102.

figurations. Then if U_B and U_C denote the potential energies at B and C respectively, then by definition

$$U_B = W_{AB},$$
$$U_C = W_{AC},$$

where W_{AB} and W_{AC} are equal, respectively, to the work done in going from A to B and from A to C. Therefore

$$U_C - U_B = W_{AC} - W_{AB} = W_{BC}, \qquad \text{(VIII)}$$

where W_{BC} equals the work done in taking the system from B to C. Thus the work done against conservative forces acting upon a system equals the increase in the potential energy of the system.

163. Isolated System. — A system which is not acted upon by external forces is called an *isolated system.* An isolated system neither gives energy to external bodies nor receives energy from them. This is an immediate result of the definition of an isolated system, because exchange of energy

presupposes work by or against external forces, which in its turn presupposes interaction with external bodies. But since no external forces are supposed to act upon the system, there cannot be interaction with external bodies or exchange of energy.

164. The Principle of the Conservation of Energy. — One of the greatest achievements of the nineteenth century was the recognition and the experimental verification of the great generalization known as the *principle of the conservation of energy*, which states that the total amount of energy of an isolated system is constant.

By means of the interaction of the different parts of an isolated system the various forms of its energy may be changed into other forms, and the distribution of the energy within the system may be altered, but the total amount of energy remains constant. In other words, *energy may be transformed or transferred but cannot be annihilated or created.*

165. Dynamical Energy. — Kinetic and potential forms of energy are called dynamical energy. The distinction between dynamical and nondynamical energy, such as heat energy, chemical energy, etc., is a matter of convenience. Heat energy may be treated as kinetic energy, but in order to do that molecules and their individual motions have to be taken into account. On the other hand chemical energy may be treated as potential energy if molecular and atomic forces can be taken into account. It is to avoid the complications of the molecular structure of bodies that these forms of energy are considered as nondynamical.

166. Conservation of Dynamical Energy. — When all the forces acting within an isolated system are conservative the interchange of energy is confined to the potential and kinetic forms of the energy of the system. Therefore applying the general principle of the conservation of energy we see that in such a system the sum of the dynamical energy remains constant, that is,

$$T + U = \text{const.} \tag{IX}$$

If T_0 and U_0 denote the initial values of T and U, then the last relation gives

$$T + U = T_0 + U_0$$

and

$$T - T_0 = -(U - U_0). \tag{X}$$

Therefore if only conservative forces act between the various parts of an isolated system, the sum of the potential and kinetic energies of the system remains constant, in other words, *the gain in the kinetic energy equals the loss in the potential energy.* Equation (X) will be called the *energy equation.*

167. Conservation of Dynamical Energy and the Law of Action and Reaction. — The principle of the conservation of dynamical energy may be obtained from the Law of Action and Reaction. In order to prove this statement consider an isolated conservative system. Suppose the configuration of the system to have changed under the action of its internal forces. Let U_0 and U be the potential energies in the initial and final configurations, respectively. Then the change in the potential energy is

$$(U - U_0).$$

During the change in the configuration of the system the positions and the velocities of the particles, which form the system, undergo changes. Therefore let s_0 and s denote the positions, and $\mathbf{v}_0$ and $\mathbf{v}$ the velocities of any particle in the initial and final configurations of the system. Further let $\mathbf{F}$ denote the resultant force which acts upon the particle. Then the change in the potential energy of the system due to the displacement of the particle from s_0 to s is

$$-\int_{s_0}^{s} F_\tau \, ds,^*$$

where F_τ is the tangential component of the force. The normal component contributes nothing to the work. There-

* Potential energy is, by definition, the work done by external forces *against* internal forces. Therefore when the change in potential energy is obtained by computing the work done *by* internal forces the result is the negative of the change in the potential energy. Hence the negative sign.

fore the total change in the potential energy of the system equals the sum of the work done, during the rearrangement, on all the particles of the system; i.e.,

$$(U - U_0) = -\Sigma \int_{s_0}^{s} F_r\, ds,$$

where the summation covers all the particles of the system. Therefore substituting the expression for F_r, which was obtained by applying the law of action and reaction to the motion of particles, we obtain

$$\begin{aligned}(U - U_0) &= -\Sigma \int_{s_0}^{s} m\frac{dv}{dt}\cdot ds \\ &= -\Sigma\, m \int_{v_0}^{v} v\, dv \\ &= -\Sigma\left(\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2\right) \\ &= -\left[\Sigma\left(\tfrac{1}{2}mv^2\right) - \Sigma\left(\tfrac{1}{2}mv_0^2\right)\right] \\ &= -(T - T_0),\end{aligned}$$

where T_0 and T are, respectively, the values of the total kinetic energy of the system in the initial and in the final configurations. Rearranging the terms of the last equation we get

$$U + T = U_0 + T_0 = \text{const.}$$

which is the principle of the conservation of dynamical energy. Therefore the principle of the conservation of dynamical energy and the law of force are not independent of each other but form two different aspects of the same universal principle.

ILLUSTRATIVE EXAMPLE.

Taking into account the variation of the gravitational attraction with the distance of a body from the center of the earth, find the potential energy of a body with respect to the surface of the earth.

Outside the earth the weight of a body varies inversely as the square of its distance from the center of the earth. Therefore denoting this variable weight by F we have

$$F = \frac{k}{r^2},$$

where k is a constant and r is the distance of the body from the center of the earth. But at the surface of the earth the weight of the body is $-mg$, therefore $F = -mg$ when $r = a$, where a is the radius of the earth. Therefore making these substitutions in the last equation we obtain

$$-mg = \frac{k}{a^2}, \text{ or } k = -mga^2.$$

Therefore $$F = -\frac{mga^2}{r^2}$$

and
$$\begin{aligned} U &= \int_a^r F\,dr \\ &= -mga^2 \int_a^r \frac{dr}{r^2} \\ &= mga^2\left(\frac{1}{a} - \frac{1}{r}\right). \end{aligned}$$

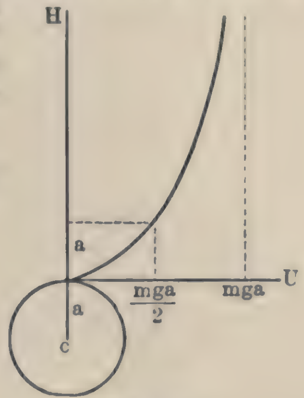


FIG. 103.

DISCUSSION. — Plotting the potential energy as abscissa and the height above the surface of the earth as ordinate we obtain the curve of Fig. 103, where the circle represents the earth.

When $r = a$, $U = 0$; as it should. When $r = \infty$, $U = mga$. Therefore mga is the maximum value of the potential energy. In the figure this is evident from the fact that the curve approaches asymptotically to the line $U = mga$. When $r = 2a$, $U = \frac{mga}{2}$. Therefore at a height of about 4000 miles the potential energy equals half its maximum value.

It will be seen from the following analysis that for small heights the potential energy may be considered to increase linearly with h, where h is the height above the surface of the earth:

$$\begin{aligned} U &= mga^2\left(\frac{1}{a} - \frac{1}{r}\right) \\ &= mga^2\left(\frac{1}{a} - \frac{1}{a+h}\right) \\ &= mga\,\frac{h}{a+h} \\ &\doteqdot mgh, \quad \text{when} \quad h \ll a.^* \end{aligned}$$

* The symbols "$\ll$" and "$\gg$" will be used to denote great inequalities. Thus "$h \ll a$" should be read "h is negligible compared with a" or "h is very small compared with a." On the other hand "$a \gg h$" should be read "a is very large compared with h."

PROBLEMS.

1. A reservoir which is 50 feet long, 40 feet wide, and 10 feet deep is full of water. Find the potential energy of the water relative to a plane 25 feet below the bottom of the reservoir.

2. A particle slides down a curve in a vertical plane and "loops the loop." Find the minimum height the starting point can have above the center of the "loop." The radius of the "loop" is 15 feet.

3. Find the least velocity with which a bullet will have to be projected from the earth so that it will never return again.

4. A uniform rod which is free to rotate about a horizontal axis is held in a horizontal position. With what angular velocity will it pass the vertical position if it is let go?

5. A cylinder of mass m and radius a is rotating about a horizontal axis, making n turns per second. How high can it raise a mass m', which is suspended from the cylinder by means of a string of negligible mass?

6. A particle, which is attached to a point by a string of negligible mass, has just enough energy to make complete revolutions in a vertical circle. Find the velocity at the highest and at the lowest points.

7. In the preceding problem show that the tension of the string is zero when the particle is at the highest point and six times the weight of the particle when it is at the lowest point.

8. A particle starts from rest at the highest point of a smooth sphere and slides down under its own weight. Where will it leave the sphere?

9. A particle which is suspended by means of a string is pulled to one side until it makes an angle α with the vertical, and then it is let go. Find the position at which the tension of the string equals the weight of the particle.

10. In the preceding problem show that the total energy remains constant during the motion of the particle. Also find the velocity at the lowest position when $\alpha = 60°$.

11. Supposing the tensile force necessary to stretch an elastic string to be proportional to the increase in length, derive an expression for the potential energy of a stretched string.

12. Derive an expression for the potential energy of a watch spring.

GENERAL PROBLEMS.

1. What should be the tractive force of a locomotive in order that it may be able to give a train of 150 tons a velocity of 45 miles per hour within one mile from the start? The resistance per ton is given in pounds by the numerical relation $R = 5 + 0.4\,v^2$, where v is the velocity in miles per hour.

2. In the preceding problem find the limiting velocity.

3. The effective horse power of a vertical water wheel is 46 and its efficiency is 70 per cent. If the head of water is 25 feet find the number of gallons of water which have to be delivered to the wheel per minute.

4. A belt running at a speed of 1500 feet per minute transmits 25 horse power. Assuming the tensile force on the tight side of the belt to be twice that on the slack side, find both tensile forces.

5. In the preceding problem find the width of the belt if the safe tensile force is 75 pounds per inch width of the belt.

6. Find the power which may be transmitted by a belt under the following conditions:

The width of the belt is 10 inches.

The pulley which the belt drives is 4 feet in diameter and makes 125 revolutions per minute.

The arc of contact subtends an angle of 150° at the center.

The coefficient of friction is 0.4.

The tensile force of the belt is not to exceed 90 pounds per inch of width.

7. In the preceding problem find the tensile force on the slack side of the belt.

8. In problem 6 suppose the arc of contact to subtend 120° at the center of the pulley.

9. In the preceding problem find the tensile force per inch width of the belt on the slack side.

10. Find the power lost due to friction in the bearings of a flywheel under the following conditions:

The journals are 6 inches in diameter and 10 inches long.

The coefficient of friction is 0.004.

The flywheel weighs 15 tons and makes 200 revolutions per minute.

The normal pressure on the bearings is constant over the surface of contact.

11. In the preceding problem suppose the vertical component of the total reaction to be constant.

12. Find the power lost due to friction in the bearings of a water turbine under the following conditions:

> The rotating system, which weighs 50 tons, is supported by a flat-end pivot bearing 10 inches in diameter.
>
> The coefficient of friction is 0.01.
>
> The turbine makes 250 revolutions per minute.

13. In the preceding problem suppose the shaft which carries the rotating system to be hollow, with an inner diameter of 6 inches and outer diameter of 12 inches.

14. In problem 12 suppose the bearing to be a hemispherical pivot with constant normal pressure.

15. In the preceding problem suppose the vertical component of the normal pressure to be constant.

CHAPTER X.

FIELDS OF FORCE AND NEWTONIAN POTENTIAL.

168. Fields of Force. — If a particle experiences a force when placed at any point of a region the region is called a *field of force*. The gravitational field of the earth, the electrical field of a charged body, and the magnetic field of a magnet are examples of fields of force.

169. Potential Energy and Fields of Force. — The potential energy of a system is due to the overlapping of the fields of force of its parts. For instance, the earth and the moon are not connected by anything material, yet they form a system which has potential energy, because they are in each other's gravitational field of force. The fact that a stretched elastic string has potential energy seems to contradict this statement, but this contradiction is only apparent. The potential energy of the stretched string is also due to the overlapping of the fields of force of its parts. In this case, however, molecules form the parts of the system.

170. The Principle of the Degradation of Potential Energy. — Consider a body which is displaced under the action of the forces of a field of force. A certain amount of work is done during the displacement. If the body is not acted upon by forces which are external to the field, then by the principle of the conservation of energy, the energy of the body remains constant during the displacement. Therefore the amount lost by one form of the energy of the body is gained by the other. The work done is the measure of the amount of the energy transformed.

The principle of the conservation of energy does not throw any light on the question, "Which form of energy is the

loser and which the gainer?" It merely states that the loss equals the gain. In order to know the direction of the transformation we have to appeal to another principle; i.e., *the principle of the degradation of potential energy*, which states:

*A body which is free to move in a field of force moves in such a way as to diminish its potential energy.**

This principle is nothing more or less than a simple statement of human experience with things that "run down." The principle states that water flows down hill under the action of gravitational forces, that a clock runs down, etc.

171. Force Experienced by a Particle in a Field of Force. — Consider a particle in a field of force. When the particle is displaced through a distance ds, under the action of the forces of the field, a certain amount of work is done which equals Fds, where **F** is the resultant force due to the field. Therefore, by the principle of the degradation of energy, the potential energy of the particle is diminished by an amount equal to Fds.

Let the rate of increase of U along the direction of the displacement be denoted by $\frac{\partial U}{\partial s}$, then $-\frac{\partial U}{\partial s} ds$ is the diminution in the potential energy. Therefore, equating the work done by the forces of the field to the diminution of the potential energy of the particle, we get

$$F\,ds = -\frac{\partial U}{\partial s} ds,$$

or

$$F = -\frac{\partial U}{\partial s}.\dagger \qquad \text{(I)}$$

* This principle may be called the dynamical version of the second law of thermodynamics.

† It should be remembered that the forces which enter into the equations $U = \int F\,ds$ and $F = -\frac{\partial U}{\partial s}$ are equal but oppositely directed. In the second equation F represents the resultant force which a particle experiences by virtue of its potential energy. On the other hand F in the definition of potential energy denotes the external force which has to be applied to the particle in

Splitting equation (I) into three component equations we have

$$\left.\begin{aligned} X &= -\frac{\partial U}{\partial x}, \\ Y &= -\frac{\partial U}{\partial y}, \\ Z &= -\frac{\partial U}{\partial z}. \end{aligned}\right\} \qquad \text{(I}'\text{)}$$

Equations (I) and (I′) state that *the force along a given direction which a particle experiences by virtue of its potential energy equals the rate of diminution of the potential energy along the given direction.*

172. Torque Experienced by a Rigid Body in a Field of Force. — Suppose the rigid body to be displaced under the action of the forces of the field through an angle $d\theta$. Then an amount of work $G\,d\theta$ is done, where **G** is the torque which the body experiences in the field. By the principles of the conservation and degradation of dynamical energy this work must come from the potential energy of the body in the field. Therefore denoting the rate of increase in the potential energy of the body, due to a rotation about a given axis, by $\frac{\partial U}{\partial \theta}$ we have

$$G\,d\theta = -\frac{\partial U}{\partial \theta}\,d\theta,$$

or

$$G = -\frac{\partial U}{\partial \theta}. \qquad \text{(II)}$$

Equation (II) states that *the torque which a rigid body experiences by virtue of its potential energy equals the rate of diminution of this energy as the body turns about the axis of the torque.*

order to overcome the forces which the particle experiences because of its position in a field of force, and thereby to bring the particle from the standard configuration to the one in which it has potential energy U.

ILLUSTRATIVE EXAMPLES.

1. Find the force which a particle placed upon a smooth inclined plane experiences by virtue of its potential energy. Also find the components of the force along the axes of a rectangular system, in which the z-axis is normal to the inclined plane and the x-axis is horizontal.

Let the origin of the axes, Fig. 104, be the position of the particle. Then if h denotes its height from the base of the inclined plane the potential energy is mgh. Therefore the force along the vertical is given by

$$F = -\frac{\partial U}{\partial h} = -\frac{\partial}{\partial h}(mgh) = -mg.$$

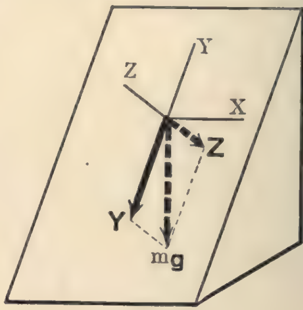


FIG. 104.

Thus the force due to the gravitational field is downwards and equals the weight of the particle. The components of the force are found by equation (I′). Thus

$$X = -\frac{\partial U}{\partial x} = -mg\frac{\partial h}{\partial x} = 0.$$

Therefore the force along the x-axis is nil.

$$Y = -\frac{\partial U}{\partial y} = -mg\frac{\partial h}{\partial y} = -mg \sin \alpha.$$

Therefore the component of the force along the plane is downwards and equals $mg \sin \alpha$.

$$Z = -\frac{\partial U}{\partial z} = -mg\frac{\partial h}{\partial z} = -mg \cos \alpha.$$

Therefore the component along the z-axis tends to move the particle normally into the plane and has a magnitude equal to $mg \cos \alpha$. The components along the x-axis and the z-axis produce no motion because X equals zero and Z is exactly balanced by the reaction of the plane. The foregoing results may be verified by finding the components of mg by the common method, i.e., by taking projections of mg along the axes.

2. A rigid body which is free to rotate about a horizontal axis is displaced through an angle θ. Find the torque due to the gravitational field of the earth.

Let A, Fig. 105, be the body, O the point where the axis pierces the plane of the paper, C the center of mass, and D its distance from the axis. Then at the displaced position the potential energy is

$$\begin{aligned} U &= mgh \\ &= mgD\,(1 - \cos\theta). \end{aligned}$$

Therefore the torque experienced by the body is

$$\begin{aligned} G &= -\frac{\partial U}{\partial \theta} \\ &= -mg\,D \sin\theta. \end{aligned}$$

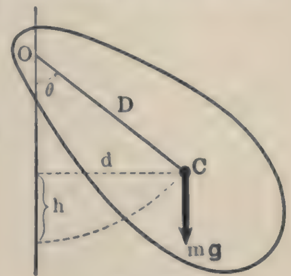


FIG. 105.

This result may be easily verified by considering the moments of the forces which act upon the body. The forces which act upon the body are the reaction of the axis and the weight of the body. The moment of the reaction is nil; therefore the resultant moment is entirely due to the weight and equals

$$G = -mg \cdot d = -mg\,D \sin\theta,$$

which is the result obtained by the other method. The negative sign is introduced to indicate the fact that the rotation is clockwise.

173. New Condition of Equilibrium. — Equations (I), (I′), and (II) provide us with a new condition for the equilibrium of conservative systems. It was shown in Chapters II and III that a system is in equilibrium when the resultant force and the resultant torque vanish.

Therefore setting F and G equal to zero in equations (I) and (II) we obtain

$$\left.\begin{aligned} \frac{\partial U}{\partial s} &= 0, \\ \frac{\partial U}{\partial \theta} &= 0, \end{aligned}\right\} \qquad \text{(III)}$$

where the differentiation in the first equation is with respect to any direction and that in the second with respect to an angle about any axis. But when equations (III) are satisfied, U has a stationary value, that is, the value of U is either a minimum, or a maximum, or a constant. Therefore the

new condition states — *in order that a conservative system be in equilibrium its potential energy must have a stationary value.*

174. Analytical Criterion of Stability. — The equilibrium of a body is said to be stable if it is not upset when the body is given a small displacement.

Potential energy is a function of coördinates, therefore we can denote the potential energy of a particle at the point (x_1, y_1, z_1) by the functional relation

$$U_1 = U\,(x_1, y_1, z_1).$$

Let us suppose the point (x_1, y_1, z_1) to be a position of equilibrium of the particle, and investigate the stability of the equilibrium. If the particle is given a displacement δx, the potential energy in the new position becomes

$$U_2 = U\,(x_1 + \delta x, y_1, z_1).$$

Expanding U_2 by Maclaurin's theorem in powers of δx we obtain

$$U_2 = U_1 + \left(\frac{\partial U}{\partial x}\right)_1 \delta x + \left(\frac{\partial^2 U}{\partial x^2}\right)_1 \frac{(\delta x)^2}{2!} + \cdot \cdot \cdot ,$$

where the subscripts after the parentheses denote that after the indicated differentiations are carried out the coördinates x, y, and z must be replaced by x_1, y_1, and z_1, which are the coördinates of the equilibrium position.

But since the particle is in equilibrium at the point (x_1, y_1, z_1)

$$\left(\frac{\partial U}{\partial x}\right)_1 = 0.$$

$$\therefore \quad U_2 - U_1 = \left(\frac{\partial^2 U}{\partial x^2}\right)_1 \frac{(\delta x)^2}{2!} + \cdot \cdot \cdot \cdot .$$

Since δx, the displacement, is small we can neglect all the terms of the right-hand member of the last equation except the first. This gives

$$U_2 - U_1 = \frac{1}{2}\left(\frac{\partial^2 U}{\partial x^2}\right)_1 (\delta x)^2.$$

Case I. — Suppose $\left(\frac{\partial^2 U}{\partial x^2}\right)_1$ to be positive. Then $U_2 - U_1$ is positive and consequently U_1 is a minimum. But according to the principle of the degradation of energy a body, which is free, moves in such a way as to diminish its potential energy. Therefore when the force which produced the displacement δx is removed the particle returns to the point (x_1, y_1, z_1), where its potential energy is a minimum. Evidently the equilibrium is *stable* in this case.

Case II. — Suppose $\left(\frac{\partial^2 U}{\partial x^2}\right)_1$ to be negative. Then $U_2 - U_1$ is negative and consequently U_1 is a maximum. Therefore if the particle is given a small displacement δx and then left to itself, it will move away from the point (x_1, y_1, z_1), where its potential energy is a maximum. In this case the equilibrium is *unstable.*

Case III. — Suppose $\left(\frac{\partial^2 U}{\partial x^2}\right)_1$ to be zero. There are three special cases to be considered:

(a) The order of the first differential coefficient which does not vanish is odd.

(b) The order of the first differential coefficient which does not vanish is even.

(c) All of the differential coefficients vanish.

It is evident that when (c) is true the potential energy of the particle has a constant value and does not change with the position of the particle. Therefore when the particle is left to itself after giving it a small displacement it will neither return to its original position nor go on changing its position. The potential energy is the same and the particle is in equilibrium at all points. In this case the equilibrium is said to be *neutral* or *indifferent.*

It may be shown that when (a) holds the equilibrium is stable. On the other hand when (b) is true the equilibrium is stable or unstable according as the first differential coefficient which does not vanish is positive or negative.

The three types of equilibrium are illustrated by the three equilibrium positions, Fig. 106, which a right cone can assume on a horizontal plane.

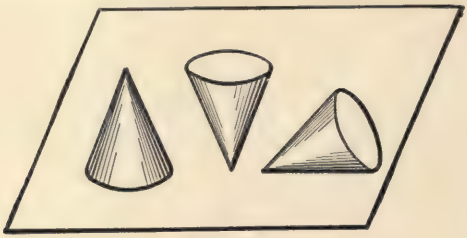


Fig. 106.

NEWTONIAN POTENTIAL.

175. Newtonian Law of Force. — The law of force between two interacting particles is called a Newtonian law of force if the particles attract or repel each other with a force which acts along the line of centers of the particles, and which varies directly as the product of the masses of the particles and inversely as the square of the distance between them. The forces between two material particles, between two small electrical charges, and between two small magnetic poles obey the Newtonian law of force. The following are the familiar forms in which the law is written for material, electrical, and magnetic particles, respectively,

$$F = -\gamma \frac{mm'}{r^2}, \quad F = -\frac{qq'}{kr^2}, \quad F = -\frac{mm'}{\mu r^2}, \tag{IV}$$

where γ, k, and μ are constants. When the interacting particles are in free space the numerical value of the constants k and μ is unity, while

$$\gamma = 6.7 \times 10^{-8} \frac{\text{cm.}^3}{\text{gm. sec.}^2}.$$

176. Newtonian Field of Force. — When the forces which act in a region obey the Newtonian law of force the region is called a *Newtonian field of force.*

177. Newtonian Potential. — The potential energy of a unit mass placed at a point of a Newtonian field is called the *potential* at that point. The standard configuration or the position of zero potential is taken to be infinitely far from the center of the field. But the potential energy of a body equals the work done in bringing the body from the position of zero potential energy, therefore the following definition is equivalent to the one just given.

The potential at a point equals the work done in bringing a unit mass from an infinite distance to that point.

178. Potential Due to a Single Particle. — Let m be the mass of the particle, U the potential energy of a particle of mass m' placed in the field of force of the first particle, r the distance between the two particles, and V the potential at the position of m' due to m. Then by the definitions of V and U

$$\left.\begin{aligned} V &= \frac{U}{m'} \\ &= \frac{1}{m'}\int_{\infty}^{r}(-F)\,dr, \end{aligned}\right\} \tag{V}$$

where F is the force experienced by m' due to the field of m.

But
$$F = -\gamma\,\frac{mm'}{r^2}.$$

Therefore
$$V = \gamma m\int_{\infty}^{r}\frac{dr}{r^2}$$

$$= -\gamma\frac{m}{r}. \tag{VI}$$

The negative sign indicates the fact that when a particle is brought to the field of another attracting particle work will be done by the particle and not by the agent which brings it. Therefore the potential due to a material particle, as we have defined it, is everywhere negative, except at infinity where it is zero. In case of electrical and magnetic masses potential is defined as the work done in bringing a unit *positive* charge, or unit positive pole, from infinity. Therefore

the potentials due to a negative charge and a negative pole are negative, while the potentials due to a positive charge and a positive pole are positive.

179. Potential Due to Any Distribution of Mass. — When the field of force or the *potential field* is due to a number of particles (material, electrical, or magnetic), the potential at a point equals the algebraic sum of the potentials due to the various particles. Thus if m_1, m_2, m_3, etc., be the masses of the particles and r_1, r_2, r_3, etc., their distances from the point considered, then the potential at the point is

$$\left.\begin{aligned} V &= -\left(\gamma\frac{m_1}{r_1} + \gamma\frac{m_2}{r_2} + \cdots\right) \\ &= -\gamma\Sigma\frac{m}{r}. \end{aligned}\right\} \tag{VII}$$

When the field is due to a continuous distribution of mass the last equation may be put in the form of an integral. Thus

$$V = -\gamma\int_0^m \frac{dm}{r}. \tag{VII$'$}$$

180. Intensity of the Field. — The intensity at any point of a potential field, or a field of force, is defined as the *force experienced by a unit mass when placed at that point.*

Let $\mathbf{H}$ denote the intensity at a point. Then, if $\mathbf{F}$ is the force experienced by a mass m' when placed at that point, we have, by definition,

$$\mathbf{H} = \frac{\mathbf{F}}{m'}, \tag{VIII}$$

and

$$\begin{aligned} H &= \frac{F}{m'} \\ &= -\frac{1}{m'}\left(\frac{\partial U}{\partial s}\right) \\ &= -\frac{\partial}{\partial s}\left(\frac{U}{m'}\right) \end{aligned}$$

$$\left.\begin{aligned} &= -\frac{\partial V}{\partial s}. \\ \text{Similarly} \qquad H_x &= -\frac{\partial V}{\partial x}, \\ H_y &= -\frac{\partial V}{\partial y}, \\ \text{and} \qquad H_z &= -\frac{\partial V}{\partial z}. \end{aligned}\right\} \tag{IX}$$

Therefore *the component, along any direction, of the intensity at any point equals the rate at which the potential diminishes at that point as one moves along the given direction.*

ILLUSTRATIVE EXAMPLES.

1. Find the expressions for potential and intensity at a point due to a spherical shell.

Let P, Fig. 107, be the point and R its distance from the center of the shell. Then taking a zone for the element of mass, as shown in the figure, we get

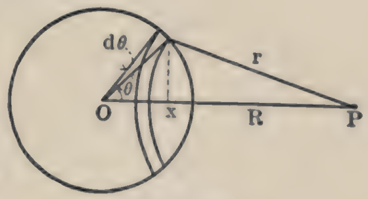


Fig. 107.

$$dm = \sigma \cdot 2\pi a \sin\theta \cdot a d\theta,$$

and
$$r = \sqrt{(R - a\cos\theta)^2 + a^2\sin^2\theta}$$
$$= \sqrt{a^2 + R^2 - 2aR\cos\theta}.$$

Therefore

$$V = -\gamma \int_0^m \frac{dm}{r}$$
$$= -\gamma\tau\, 2\pi a^2 \int_0^\pi \frac{\sin\theta\, d\theta}{\sqrt{a^2 + R^2 - 2aR\cos\theta}}$$
$$= -\frac{\gamma\tau\, 2\pi a}{R}\left[(a^2 + R^2 - 2aR\cos\theta)^{\frac{1}{2}}\right]_0$$
$$= -\frac{\gamma\tau\, 2\pi a}{R}\left[(a^2 + 2aR + R^2)^{\frac{1}{2}} - (a^2 - 2aR + R^2)^{\frac{1}{2}}\right].$$

There are two different cases which have to be considered separately.

(a) Point Outside the Sphere. — In this case $R > a$. Therefore the expression for the potential may be put in the form

$$
\begin{aligned}
V &= -\frac{\gamma\tau\, 2\,\pi a}{R}[(a+R)-(R-a)] \\
&= -\gamma\frac{\tau 4\,\pi a^2}{R} \\
&= -\gamma\frac{m}{R}.
\end{aligned}
$$

Therefore outside the shell the potential is the same as if the mass of the shell were concentrated at its center.

(b) Point Within the Sphere. — In this case $R < a$. Therefore

$$
\begin{aligned}
V &= -\frac{\gamma\tau\, 2\,\pi a}{R}[(a+R)-(a-R)] \\
&= -\gamma\frac{\tau\, 4\,\pi a^2}{a} \\
&= -\gamma\frac{m}{a}.
\end{aligned}
$$

Therefore within the shell the potential is constant and equals that at the surface.

If H denotes the intensity of the field due to the shell, then

$$
\begin{aligned}
H &= -\frac{\partial V}{\partial R} \\
&= -\gamma\frac{m}{R^2} \quad \text{when } R > a. \\
&= 0 \quad \text{when } R < a.
\end{aligned}
$$

Therefore the shell attracts a particle which is outside with the same force as if all of its mass were concentrated at its center. On the other hand the shell exerts no force on a particle which is within the shell. The distribution of V and H in the field are represented graphically in Fig. 108, where curve (I) represents the potential and (II) the intensity.

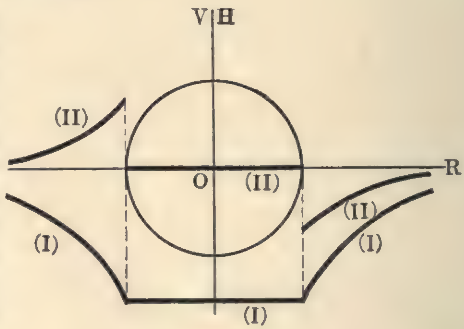


Fig. 108.

2. Find the expressions for the potential and the intensity due to a solid spherical mass.

There are two cases which have to be considered separately.

(a) Point Outside the Sphere. — Consider the sphere to be made of concentric shells of thickness $d\rho$. Then, since the point is outside every one of these shells the potential due to any one of the shells is, according to the results of the last problem,

$$dV = -\gamma\frac{dm}{R},$$

where dm is the mass of the shell and R the distance of the point from the center. Hence the potential due to all the shells in the sphere is

$$V = -\gamma\int_0^m \frac{dm}{R}$$

$$= -\gamma\frac{m}{R},$$

where m is the mass of the sphere. Therefore the potential at a point outside of a sphere is the same as that due to a particle of equal mass placed at the center.

(b) Point Within the Sphere. — In this case we divide the sphere into two parts by means of a concentric spherical surface which passes through the point. Then the potential due to that part of the sphere which is within the spherical surface is obtained by the result of case (a). Thus if m_1 denotes the mass of this part of the sphere and V_1 its potential, then

$$V_1 = -\gamma\frac{m_1}{R} = -\frac{4}{3}\pi\gamma\tau R^2.$$

In order to find the potential due to the rest of the sphere suppose it to be divided into a great number of concentric spherical shells. Then since every one of the shells contains the point the potential due to any one of them is

$$dV_2 = -\gamma\frac{dm}{\rho} = -4\pi\gamma\tau\rho\, d\rho,$$

where dm is the mass, ρ the radius, and $d\rho$ the thickness of the shell. Therefore the potential due to all the shells having radii between R and a is

$$V_2 = -4\pi\gamma\tau\int_R^a \rho\, d\rho$$

$$= -2\pi\gamma\tau\,(a^2 - R^2).$$

Therefore the potential due to the entire sphere is

$$
\begin{aligned}
V &= V_1 + V_2 \\
&= -\frac{2\pi\gamma\tau}{3}(3a^2 - R^2) \\
&= -\gamma m \frac{3a^2 - R^2}{2a^3}.
\end{aligned}
$$

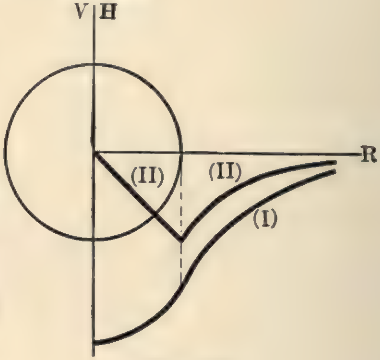


FIG. 109.

When R is plotted as abscissa and V as ordinate the distribution of the potential is given by a curve similar to (I) of Fig. 109.

Now consider the intensity at a point in the field of the sphere.

(a) POINT OUTSIDE THE SPHERE.

$$
\begin{aligned}
H &= -\frac{\partial V}{\partial R} \\
&= -\gamma \frac{m}{R^2}.
\end{aligned}
$$

Therefore the distribution of the field intensity outside of the sphere is the same as that due to a particle placed at the center.

(b) POINT WITHIN THE SPHERE.

$$
\begin{aligned}
H &= -\frac{\partial V}{\partial R} \\
&= -\gamma \frac{m}{a^3} R.
\end{aligned}
$$

Therefore within the sphere the distribution of the field intensity obeys the harmonic law; i.e., the intensity varies directly as the distance from the center. In Fig. 109, curve (II) gives the distribution of the intensity of the field.

PROBLEMS.

1. Find the potential and the field intensity due to a hollow sphere at a point (1) outside, (2) within the hollow part, and (3) in the solid part of the sphere.

2. Find the potential and the field intensity due to a circular disk of negligible thickness at a point on its axis.

3. Find the potential and the field intensity due to a straight wire of length l and mass m at a point on the axis of the wire. The cross-section of the wire is negligible.

4. Find the potential and the field intensity due to a straight circular rod at a point on its axis.

5. Show that problems 2 and 3 are special cases of problem 4.

6. Find the magnetic potential and the field intensity due to a cylindrical magnet at a point on its axis; suppose the magnetism to be distributed at the ends only.

7. Find the potential and the field intensity due to two spherical charges at a point equidistant from centers of the two charges.

8. Find the potential and the field intensity due to a right cone at a point on its axis.

9. A uniform solid sphere is cut in two by a diametral plane. Show that the gravitational force between the two parts will be $\frac{3}{16}\gamma\frac{m}{a^2}$, where m is the mass of the sphere, a the radius, and γ the gravitational constant.

10. Show that if any two points on the surface of the earth were joined by a straight and smooth tunnel a particle would traverse it in about 42.5 minutes.

11. Two spheres of masses m and m' attract each other with a force, $F = \gamma\frac{mm'}{r^n}$, where γ is a constant and r is the distance between the centers. Taking the configuration when the spheres are in contact to be that of zero potential energy, find their potential energy when the centers are separated by a distance D. The radii of the spheres are a and b.

12. In the preceding problem suppose the spheres to repel each other with the same law of force and take the configuration when the spheres are separated by an infinite distance to be that of zero potential energy.

13. Find the potential due to a small magnet at a point whose distance is large compared with the length of the magnet.

14. In the preceding problem find the components of the intensity of the field along and at right angles to the line joining the point to the magnet. Also find the total intensity and its direction.

CHAPTER XI.

UNIPLANAR MOTION OF A RIGID BODY.

181. Angular Kinetic Reaction. — It will be remembered that in considering the equilibrium of rigid bodies the Law of Action and Reaction was divided into the following two sections:

To every linear action there is an equal and opposite linear reaction, or, the sum of all the linear actions to which a body or a part of a body is subject at any instant vanishes.

$$\Sigma \mathbf{A}_l = 0. \qquad (A_l)$$

To every angular action there is an equal and opposite angular reaction, or, the sum of all the angular actions to which a body or a part of a body is subject at any instant vanishes.

$$\Sigma \mathbf{A}_a = 0. \qquad (A_a)$$

In Chapter VI the first section of the law was applied to particles in motion; but in order to do this the meaning of the terms "linear action" and "linear reaction" was enlarged so as to include linear kinetic reactions as well as forces. In the present chapter the second section of the law will be applied to the motion of rigid bodies; but before doing this we must introduce another form of kinetic reaction, which we will call *angular kinetic reaction.* If we replace in the second section of the law the terms "angular action" and "angular reaction" by the terms "torque" and "angular kinetic reaction," we obtain the following form which is directly applicable to problems of rotation:

The sum of all the torques acting upon a rigid body plus the angular kinetic reaction equals zero, or the

resultant torque equals and is oppositely directed to the angular kinetic reaction.

Resultant torque = −(angular kinetic reaction). (I)

In order to understand the nature of the angular kinetic reaction consider the following experiment: If we try to rotate a flywheel, which is free to move about a horizontal axis, by pulling at one end of a string which is wound around the axle, Fig. 110, we find that the greater the angular velocity which we want to impart in a given interval of time the harder we must pull at the string. But since the pull of the string and the reaction of the bearings form a couple and since the increase in the angular velocity per unit time means angular acceleration, we conclude that a torque must be applied to the flywheel in order to impart to it an angular acceleration, and that the greater the acceleration desired the greater must the torque be. Evidently the torque which we apply to the flywheel expends itself in overcoming certain reactions. The resisting torque due to the friction between the axle and its bearings and between the surface of the flywheel and the surrounding air must be overcome. But if we gradually diminish this resisting torque by reducing the friction we observe that the torque which must be applied, in order to give the flywheel a certain angular acceleration, tends towards a constant value different from zero. In other words even if all the resisting torques due to friction were eliminated we would have to apply a torque of definite magnitude in order to give the flywheel a desired angular acceleration; that is, the flywheel resists torques which impart to it an angular acceleration. This resistance to angular acceleration is the *angular kinetic reaction*.

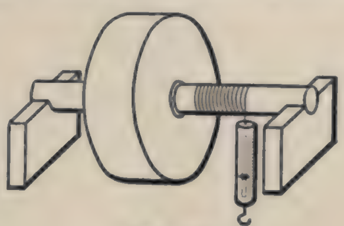


FIG. 110.

182. Experimental Definition of Moment of Inertia. — If in the experiment of the preceding section all frictional forces and torques are eliminated and then torques of different magnitudes are applied to the flywheel, it will be found that the torques are proportional to the angular accelerations produced; that is, if G_1, G_2, etc., denote the torques obtained by multiplying the pull of the string by the radius of the axle and γ_1, γ_2, etc., the corresponding angular acceleration, then we shall find that the following relations hold:

$$\frac{G_1}{\gamma_1} = \frac{G_2}{\gamma_2} = \frac{G_3}{\gamma_3} = \cdot \cdot \cdot = I, \tag{II}$$

where I is a constant which depends only upon the rotating system. In fact, as will be shown in § 186, it is nothing more or less than the moment of inertia of the rotating system. We have, therefore, the following definition for the moment of inertia of a body, in addition to the analytical definition given in Chapter VII:

*The moment of inertia of a body about a given axis is a constant of the body, relative to the given axis, which equals the quotient of the torque applied by the angular acceleration obtained; both being referred to the given axis.**

183. Measure of Angular Kinetic Reaction. — It is evident from equation (II) that G_1, G_2, etc., which measure the angular kinetic reactions of the flywheel for the accelerations γ_1, γ_2, etc., are proportional to these accelerations. Therefore the angular kinetic reaction of a body varies directly with the angular acceleration imparted. If, on the other hand, a number of bodies of different moments of inertia are given the same angular acceleration, it is found that the kinetic reactions are proportional to the moments of inertia; that is,

$$\frac{G_1}{I_1} = \frac{G_2}{I_2} = \cdot \cdot \cdot = \gamma, \tag{III}$$

* Note the striking similarity between this definition of moment of inertia and the definition of mass given in § 94.

where γ is the common angular acceleration. Therefore the angular kinetic reaction varies directly as the product of the moment of inertia by the angular acceleration,

$$\text{angular kinetic reaction} = kI\gamma,$$

where k is the constant of proportionality. When all the magnitudes involved in the last equation are measured in the same system of units k becomes unity.

Introducing this simplification in the last equation and putting it into vector notation we have

$$\textbf{angular kinetic reaction} = -I\boldsymbol{\Upsilon}. \qquad \text{(IV)}$$

The negative sign indicates the fact that the direction of the angular kinetic reaction is opposed to that of the angular acceleration.

184. Torque Equation. — Combining equations (I) and (IV) and denoting the resultant torque by **G** we obtain

$$\left.\begin{aligned} \mathbf{G} &= I\boldsymbol{\Upsilon}, \\ &= I\dot{\boldsymbol{\omega}}. \end{aligned}\right\} \qquad \text{(V)}$$

The last equation, which will be called the *torque equation*, states that the resultant torque about any axis equals the product of the moment of inertia by the angular acceleration and has the same direction as the angular acceleration.

185. The Two Definitions of Moment of Inertia. — In order to show that the constant, I, of equation (II) and the moment of inertia defined by equation (II) of page 152 are the same magnitude, consider the motion of the rigid body A, Fig. 111, about a fixed axis through the point O, perpendicular to the plane of the paper. Let $d\mathbf{F}$ be the resultant force acting upon an element of mass dm, that is, the vector sum of the forces due to external fields of force and the forces due to the connection of dm with the rest of the body. Then

$$dF = dm\frac{dv}{dt}$$

is the force equation for the element of mass.

The linear acceleration varies from point to point, but the angular acceleration is the same for all the elements. Therefore the discussion of the problem becomes simpler if we replace the linear acceleration by the angular acceleration. This may be done by taking the moments of the forces about the axis. Since dm can move only in a direction perpendicular to the line r, the resultant force $d\mathbf{F}$ must be perpendicular to r. Therefore the magnitude of the moment $d\mathbf{G}$, due to $d\mathbf{F}$, is

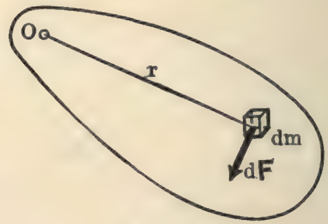


FIG. 111.

$$dG = r\,dF$$
$$= r\,dm\frac{dv}{dt}$$
$$= r^2\,dm\frac{d\omega}{dt}, \qquad (v = r\omega).$$

Therefore the resultant torque acting upon the body, or the sum of the moments due to the forces acting upon all the particles of the body, is

$$G = \int_0^m r^2\,dm\frac{d\omega}{dt}$$
$$= \dot{\omega}\int_0^m r^2\,dm.$$

But by equation (V) $G = I\dot{\omega}$. Therefore

$$I = \int_0^m r^2\,dm,$$

which is the definition of the moment of inertia given in Chapter VII.

186. Comparison. — There is a perfect analogy between motion of pure translation and motion of pure rotation. This is clearly brought out in the following lists of the magnitudes involved in the two types of motion:

Magnitudes involved in motion of translation.	Their analogues in motion of rotation.
$\mathbf{s}$, linear displacement.	$\boldsymbol{\theta}$, angular displacement.
$\mathbf{v}$, linear velocity.	$\boldsymbol{\omega}$, angular velocity.
$\dot{\mathbf{v}}$, linear acceleration.	$\dot{\boldsymbol{\omega}}$, angular acceleration.
m, linear inertia or mass.	I, angular inertia or moment of inertia.
$-m\dot{\mathbf{v}}$, linear kinetic reaction.	$-I\dot{\boldsymbol{\omega}}$, angular kinetic reaction.
$m\mathbf{v}$, linear momentum.*	$I\boldsymbol{\omega}$, angular momentum.†
$\frac{1}{2}mv^2$, kinetic energy of translation.	$\frac{1}{2}I\omega^2$, kinetic energy of rotation.
$\mathbf{F} = m\dot{\mathbf{v}}$, force equation.	$\mathbf{G} = I\dot{\boldsymbol{\omega}}$, torque equation.
$W = \int_0^s F\,ds$, work done by a force.	$W = \int_0^\theta G\,d\theta$, work done by a torque.
$\mathbf{L} = \int_0^t \mathbf{F}\,dt$, linear impulse.*	$\mathbf{H} = \int_0^t \mathbf{G}\,dt$, angular impulse.†

* Discussed in Chapter XII. † Discussed in Chapter XIII.

187. Torque and Energy Methods. — The equations of motion of a rigid body may be obtained in two ways, one of which will be called the *torque method* and the other the *energy method.*

Torque method: First, find the resultant torque and substitute it in the torque equation.

Second, integrate the torque equation in order to find the integral equations of the motion.

Energy method: First, equate the change in the potential energy to the change in the kinetic energy.

Second, differentiate the energy equation, thus obtained, with respect to the time. This gives the torque equation.

Third, proceed as in the torque method.

The energy method is advantageous in complicated problems, but not in simple ones.

ILLUSTRATIVE EXAMPLES

ON THE MOTION OF A RIGID BODY ABOUT A FIXED AXIS.

Discuss the motion of a rigid body, which is free to rotate about a fixed axis, under the action of a constant torque.

Suppose the body to be the flywheel of Fig. 110. Let the constant torque be supplied by a constant force **F** applied at the free end of the string which is wound around the axle. The tensile force of the string and the reaction of the bearings form a couple, the torque of which equals the moment of the force **F** about the axis of rotation. Therefore

$$G = Fa,$$

where a is the radius of the axle. Substituting this value of G in the torque equation we obtain

$$I\frac{d\omega}{dt} = Fa,$$

or

$$\frac{d\omega}{dt} = \frac{Fa}{I} = \gamma = \text{const.}$$

Integrating the last equation we get

$$\omega = \gamma t + c.$$

Let $\omega = \omega_0$ when $t = 0$, then $c = \omega_0$. Therefore

$$\omega = \omega_0 + \omega t, \tag{1}$$

or

$$\frac{d\theta}{dt} = \omega_0 + \omega t.$$

Integrating again

$$\theta = \omega_0 t + \tfrac{1}{2}\omega t^2 + c'.$$

Let $\theta = 0$ when $t = 0$, then $c' = 0$. Therefore

$$\theta = \omega_0 t + \tfrac{1}{2}\omega t^2. \tag{2}$$

Eliminating t between equations (1) and (2)

$$\omega^2 = \omega_0^2 + 2\gamma\theta.^* \tag{3}$$

Energy Method. — The increase in the kinetic energy due to the action of **F** is

$$T - T_0 = \tfrac{1}{2}I\omega^2 - \tfrac{1}{2}I\omega_0^2.$$

The diminution in the potential energy of the system which supplies the force **F** equals the work done by **F**. Therefore

$$-(U - U_0) = Fs,$$

* Compare equations (1), (2), and (3) with the corresponding equations of p. 113.

where s is the length of the string which is unwound. Substituting these in the energy equation we obtain

$$\tfrac{1}{2} I\omega^2 - \tfrac{1}{2} I\omega_0^{\,2} = Fs.$$

$$\therefore \quad \omega^2 = \omega_0^{\,2} + \frac{2F}{I} s$$

$$= \omega_0^{\,2} + 2\gamma\theta, \quad (s = a\theta)$$

which is the equation (3) obtained by the torque method. Differentiating the last equation we have

$$2\omega \frac{d\omega}{dt} = 2\gamma\omega$$

and

$$\frac{d\omega}{dt} = \gamma,$$

which is the equation obtained by the torque method; therefore the rest of the problem is identical with that given by the torque method.

2. A flywheel rotates about a horizontal axis under the action of a falling body, which is suspended by means of a string wound around the axle of the flywheel. Discuss the motion, neglecting the mass of the string.

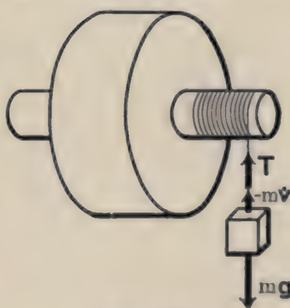


FIG. 112.

Let I = the moment of inertia of the rotating system.
m = the mass of the falling body.
a = the radius of the axle.
T = the tensile force of the string.

TORQUE METHOD. — Taking the moments about the axis of rotation we have

$$G = Ta$$

for the resultant torque. Therefore

$$I\dot{\omega} = Ta$$

is the torque equation. But considering the forces acting upon the falling body we get

$$m\dot{v} = mg - T.$$

Hence

$$I\dot{\omega} = Ta$$
$$= (mg - m\dot{v})\, a$$
$$= m\,(g - a\dot{\omega})\, a.$$

ENERGY METHOD.—Suppose the flywheel to start from rest, and let h denote the distance covered by the body during its fall. Then the energy equation gives

$$\tfrac{1}{2} I\omega^2 + \tfrac{1}{2} mv^2 = mgh.$$

Differentiating with respect to t,

$$I\omega\dot{\omega} + mv\dot{v} = mg\dot{h}.$$

But $v = a\omega$ and $\dot{h} = v$, therefore

$$I\omega\dot{\omega} + ma^2\omega\dot{\omega} = mga\omega,$$

or

$$I\dot{\omega} = m\,(g - a\dot{\omega})\, a.$$

Thus we have from either method

$$\dot{\omega} = \frac{ma}{I + ma^2} g,$$

$$\dot{v} = \frac{ma^2}{I + ma^2} g.$$

It is evident from the last two equations that both the linear and the angular accelerations are constant. Therefore the equations of motion are

$$v = \frac{ma^2}{I + ma^2} gt, \qquad \omega = \frac{ma}{I + ma^2} gt,$$

$$s = \frac{1}{2} \frac{ma^2}{I + ma^2} gt^2, \qquad \theta = \frac{1}{2} \frac{ma}{I + ma^2} gt^2,$$

$$v^2 = 2 \frac{ma^2}{I + ma^2} gh, \qquad \omega^2 = 2 \frac{ma}{I + ma^2} g\theta.$$

Discussion. — When $I \ll ma$, then $\dot{v} \doteq g$, and the motion of the suspended body is about the same as that of a freely falling body. When $I \gg ma$ then $\dot{v} \doteq 0$. Therefore the velocity of the falling body changes very slowly.

3. A uniform rectangular trapdoor, which is held in a vertical position, is allowed to fall. Supposing the hinges to be smooth and horizontal, find the expression for the angular velocity at any instant of the motion.

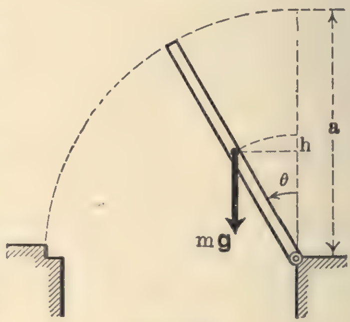


Fig. 113.

Torque Method. — The torque on the door is due to the action of its weight and the reaction of the hinges. Therefore

$$G = mg \cdot \frac{a}{2} \sin\theta.$$

Putting this value of G in the torque equation we get

$$I\dot{\omega} = \frac{mga}{2} \sin\theta.$$

Energy Method. — In turning through an angle θ the door acquires a kinetic energy of $\frac{1}{2} I\omega^2$ and loses from its potential energy an amount equal to mgh. Therefore the energy equation gives

$$\frac{1}{2} I\omega^2 = mgh = mg\frac{a}{2}(1 - \cos\theta).$$

Differentiating with respect to the time

$$I\omega\dot{\omega} = \frac{mga}{2} \sin\theta \cdot \dot{\theta},$$

or

$$I\dot{\omega} = \frac{mga}{2} \sin\theta.$$

$$\therefore \quad \dot{\omega} = \frac{mga}{2I}\sin\theta,$$

or
$$\frac{d^2\theta}{dt^2} = \frac{3g}{2a}\sin\theta. \quad \left(I = \frac{ma^2}{3}.\right)$$

Multiplying both sides of the last equation by $2\frac{d\theta}{dt}dt$, and integrating we obtain

$$\left(\frac{d\theta}{dt}\right)^2 = -\frac{3g}{a}\cos\theta + c.$$

But $\frac{d\theta}{dt} = 0$ when $\theta = 0$, therefore $c = \frac{3g}{a}$. Hence

$$\omega^2 = \frac{3g}{a}(1 - \cos\theta).$$

Discussion. — It will be observed that this result is already given by the energy equation.

When $\theta = \frac{\pi}{2}$, $\omega^2 = \frac{3g}{a}$, therefore the door strikes the floor with an angular velocity of $\sqrt{\frac{3g}{a}}$. Thus the greater a the less the angular velocity with which the door strikes the floor. On the other hand the linear velocity with which the end of the door strikes the floor increases with a, since

$$v^2 = 3ag(1 - \cos\theta)$$
$$= 3ag, \text{ when } \theta = \frac{\pi}{2}.$$

PROBLEMS.

1. Discuss the motion of the falling bodies in Atwood's machine, supposing the pulley to rotate without slipping.

2. In the problem discussed in the first illustrated example take into account the resistance of the air, supposing the resistance to be proportional to the angular velocity of the wheel.

3. A flywheel which is making 400 revolutions per minute and which is subject to a constant torque of 50 pounds-foot comes to rest after making 1500 revolutions. Find the moment of inertia of the wheel, the angular acceleration and the time taken in coming to rest. The angular acceleration is supposed to be constant.

4. A flywheel which is subject to a constant torque of 5000 dynes-centimeter starts from rest and makes 2000 revolutions in 4 minutes. Find the angular acceleration and the moment of inertia.

5. In the second illustrative problem suppose there is a resistance to the motion of the falling body proportional to its velocity.

6. A flywheel making 400 revolutions per minute is brought to rest in 3 minutes by means of friction brakes applied to it. Find the angular inertia of the wheel and axle if the total brake-shoe force applied is 500 pounds and the diameter of the flywheel is 10 feet.

7. In the preceding problem find the total number of revolutions made after the brakes were applied.

8. A flywheel is brought to rest by means of brakes applied at the axle. If the combined angular inertia of the flywheel and the axle is 50,000 gm. cm.^{2} and the diameter of the axle 20 cm., find the force which must be applied on the brakes in order to bring the flywheel to rest within 5 minutes, the initial angular velocity being 30 radians per sec.

9. A flywheel is stopped by fluid friction. The resisting torque due to the friction is proportional to the angular velocity. Discuss the motion.

10. The flywheel of a gyroscope is rotated by applying a force to a string wound around the axle. Discuss the motion, supposing the tension of the string to be proportional to the length of the string unwound.

MOTION OF A RIGID BODY ABOUT INSTANTANEOUS AXES.

188. Uniplanar Motion. — It was shown on p. 31 that uniplanar motion may be considered as a motion of pure rotation at each instant of the motion. Since the torque and energy equations hold good at each instant of the motion they can be applied to a rigid body in uniplanar motion as if the instantaneous axis were fixed at the instant considered. Therefore uniplanar motion may be discussed in the same way as motion about a fixed axis.

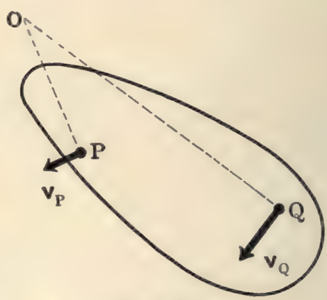


FIG. 114.

189. Instantaneous Axis. — If at any instant the velocities of two points of a rigid body are known the position of the instantaneous axis may be found in the following manner: Let P and Q, Fig. 114, be two points which lie in a plane parallel to the guide plane, and the velocities of which are

not parallel; further let $\mathbf{v}_P$ and $\mathbf{v}_Q$ be the velocities. Draw the line PO perpendicular to $\mathbf{v}_P$ in a plane parallel to the guide plane; also draw QO perpendicular to $\mathbf{v}_Q$ in the same plane. Then the instantaneous axis passes through O, the point of intersection, and is perpendicular to the plane POQ.

ILLUSTRATIVE EXAMPLES.

1. Discuss the motion of a uniform circular cylinder which rolls down a rough inclined plane without slipping.

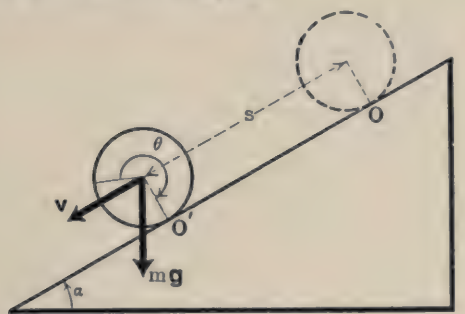


Fig. 115.

Let m = the mass of the cylinder.

I = the moment of inertia of the cylinder about the element of contact.

a = the radius of the cylinder.

v = the velocity of the axis of the cylinder.

ω = the angular velocity of the cylinder.

Torque Method. — The torque is due to the weight of the cylinder and the reaction of the plane. It equals the moment of the weight about the element of contact. Therefore

$$G = mga \sin \alpha.$$

Substituting this value of G in the torque equation we have

$$I\dot{\omega} = mga \sin \alpha.$$

Energy Method. — In moving through a distance s along the plane the potential energy of the cylinder is diminished by an amount equal to

$$mgh = mgs \sin \alpha.$$

Therefore the energy equation gives

$$\tfrac{1}{2} I\omega^2 - \tfrac{1}{2} I\omega_0^2 = mgs \sin \alpha.$$

Differentiating with respect to the time

$$I\omega\dot{\omega} = mg\dot{s} \sin \alpha.$$

$$\therefore \quad I\dot{\omega} = mga \sin \alpha.$$

Therefore $$\dot{\omega} = \frac{2}{3a} g \sin \alpha,$$

and $$\dot{v} = \frac{2}{3} g \sin \alpha.$$

Thus both the linear and the angular accelerations of the cylinder are constant. Therefore the equations of the motion are the following:

$$v = v_0 + \frac{2}{3} gt \sin \alpha, \qquad \omega = \omega_0 + \frac{2}{3a} g \sin \alpha,$$

$$s = v_0 t + \frac{1}{3} gt^2 \sin \alpha, \qquad \theta = \omega_0 t + \frac{1}{3a} gt^2 \sin \alpha,$$

$$v^2 = v_0^2 + \frac{4}{3} gs \sin \alpha. \qquad \omega^2 = \omega_0^2 + \frac{4}{3a} g\theta \sin \alpha.$$

2. A wheel moves down an inclined groove with its axle rolling along the groove without slipping. Discuss the motion.

Let a = the radius of the axle.
b = the radius of the wheel.
m' = the mass of that part of the axle which projects out from the wheel.
m = the mass of the rest of the moving system.
$M = m + m'$.

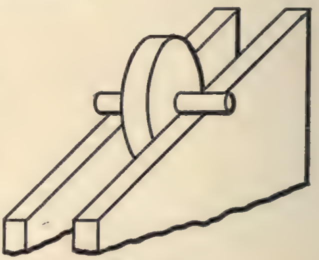


FIG. 116.

Suppose the wheel to be a solid disk with a thickness equal to half the total length of the axle. Then if both the wheel and the axle are of the same material the relation $\dfrac{m'}{m} = \dfrac{a^2}{b^2}$ holds. Therefore

$$m' = \frac{a^2}{a^2 + b^2} M \quad \text{and} \quad m = \frac{b^2}{a^2 + b^2} M.$$

TORQUE METHOD.—Considering the moments about the element of contact we obtain the following for the torque equation:

$$I\dot{\omega} = Mga \sin \alpha,$$
$$(I_c + Ma^2)\, \dot{\omega} = Mga \sin \alpha,$$
$$(I_c + Ma^2)\, \dot{v} = Mga^2 \sin \alpha,$$

where I_c denotes the moment of inertia of the moving system about its own axis.

ENERGY METHOD. — Supposing the wheel to start from rest we obtain

$$\tfrac{1}{2} Mv^2 + \tfrac{1}{2} I_c\omega^2 = Mgs \sin \alpha,$$
$$\tfrac{1}{2}\left(Mv^2 + I_c \frac{v^2}{a^2}\right) = Mgs \sin \alpha,$$
$$\tfrac{1}{2} (I_c + Ma^2)\, v^2 = Mga^2 s \sin \alpha,$$
$$(I_c + Ma^2)\, v\dot{v} = Mga^2 \dot{s} \sin \alpha,$$
$$(I_c + Ma^2)\, \dot{v} = Mga^2 \sin \alpha.$$

$$\therefore \quad \dot{v} = \frac{Ma^2}{I_c + Ma^2} g \sin \alpha,$$

$$\therefore \quad \omega = \frac{Ma}{I_c + Ma^2} g \sin \alpha.$$

Thus both the linear acceleration and the angular acceleration are constant. Therefore the equations of motion may be obtained as in the preceding problem.

DISCUSSION.

$$I_c = \frac{m'a^2}{2} + \frac{mb^2}{2}$$

$$= M \frac{a^4 + b^4}{2(a^2 + b^2)}.$$

Substituting this value of I_c in the expression for $\dot{v}$ we get

$$\dot{v} = \frac{2a^2(a^2 + b^2)}{2a^4 + (a^2 + b^2)^2} g \sin \alpha.$$

Case I.—Let $b = a$, then $\dot{v} = \frac{2}{3} g \sin \alpha$, which is the acceleration of a cylinder rolling down an inclined plane.

Case II.—Let $b \ll a$, then $\dot{v} \doteq \frac{2}{3} g \sin \alpha$, as in case I.

Case III.—Let $b \gg a$, then $\dot{v} \doteq \frac{2a^2}{b^2} g \sin \alpha$. Thus by reducing the radius of the axle we can reduce the acceleration, theoretically at least, as much as we please. The reason for this fact becomes clear when we consider the relative proportions of the potential energy which are transformed into kinetic energy of translation and kinetic energy of rotation.

3. In Fig. 117 the larger circle represents a cylinder of mass M which rolls along a rough horizontal table, under the action of a falling body of mass m. The right-hand end of the ribbon, which connects the falling body with the cylinder, is wound around the latter so that it is unwound as the motion goes on. The pulley over which the ribbon slides is smooth. Discuss the motion, supposing the mass and the thickness of the ribbon to be negligible.

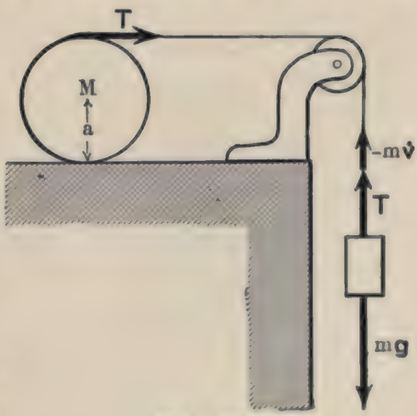


FIG. 117.

Torque Method. — The cylinder is acted upon by four forces — its weight $M\mathbf{g}$, the normal reaction $\mathbf{N}$, the frictional reaction $\mathbf{F}$, and the tensile force of the string $\mathbf{T}$. Taking the moments about the element of contact we obtain

$$G = T \cdot 2a,$$

where a is the radius of the cylinder. The other forces do not have moments about the element of contact. But considering the motion of the falling body we find that

$$m\dot{v} = mg - T,$$

where $\dot{v}$ is the acceleration of the falling body. Therefore

$$G = m(g - \dot{v}) \cdot 2a.$$

Substituting this value of G in the torque equation,

$$I\dot{\omega} = 2am(g - \dot{v}),$$

where I is the moment of inertia of the cylinder about the element of contact, and $\dot{\omega}$ the angular acceleration. But since the highest element of the cylinder has the same linear velocity as the ribbon and the falling body, we have $2a\omega = v$, and consequently $\dot{\omega} = \dfrac{\dot{v}}{2a}$. Making this substitution in the torque equation

$$\frac{I\dot{v}}{2a} = 2am(g - \dot{v}),$$

or

$$\dot{v} = \frac{4a^2m}{I + 4a^2m} \cdot g$$

$$= \frac{m}{m + \frac{3}{8}M}\, g. \qquad (I = \tfrac{3}{2}Ma^2.)$$

Energy Method. — Supposing the initial velocities to be zero and equating the gain in the kinetic energy of the system to the loss in potential energy we have

$$\tfrac{1}{2}I\omega^2 + \tfrac{1}{2}mv^2 = mgh,$$

where h is the distance fallen through by the body. Differentiating the last equation with respect to the time,

$$I\omega\dot{\omega} + mv\dot{v} = mg\dot{h}.$$

But $\dot{h} = v$, $\omega = \dfrac{v}{2a}$, and $\dot{\omega} = \dfrac{\dot{v}}{2a}$. Making these changes and solving for $\dot{v}$ we obtain

$$\dot{v} = \frac{4a^2m}{I + 4a^2m}\, g \qquad \dot{\omega} = \frac{2am}{I + 4a^2m}\, g,$$

$$= \frac{m}{m + \frac{3}{8}M}\, g, \qquad = \frac{m}{2a(m + \frac{3}{8}M)}\, g,$$

which are the expressions obtained by the torque method.

Discussion. — It is evident that both $\dot{v}$ and $\dot{\omega}$ are constant. Therefore the motions of both the cylinder and the falling body are uniformly accelerated, the one in rotation, the other in translation.

When m is negligible compared with M, $\dot{v}$ is very small and consequently the motion very slow. When m is very large compared with M, $\dot{v}$ is practically equal to g, hence the body falls almost freely.

The linear acceleration of the axis of the cylinder equals one-half that of the falling body. The linear accelerations of the cylinder and of the falling body depend upon the radius of the cylinder only indirectly, i.e., through the mass of the cylinder.

4. A circular hoop is projected along a rough horizontal plane with a linear velocity v_0 and an angular velocity ω_0. Discuss the motion.

The hoop is acted upon by two forces, namely, its weight and the reaction of the plane. The latter may be resolved, as usual, into its normal component $\mathbf{N}$ and its frictional component $\mathbf{F}$. Then the force equation gives

$$m\frac{dv}{dt} = \pm F \tag{1}$$

for the horizontal direction and

$$0 = N - mg \tag{2}$$

for the vertical direction. On the other hand the torque equation gives

$$I_c\frac{d\omega}{dt} = \mp Fa, \tag{3}$$

where a is the radius of the hoop and I_c its moment of inertia about its own axis. The double sign indicates the fact that the direction of $\mathbf{F}$ changes with the direction in which slipping takes place at the point of contact. Denoting the coefficient of friction at the point of contact by μ we have

$$\begin{aligned} F &= \mu N, \\ &= \mu mg \qquad \text{[by equation (2)].} \end{aligned}$$

Making this substitution in equations (1) and (3) and replacing I_c in equation (3) by its value we obtain

$$\frac{dv}{dt} = \pm \mu g \tag{4}$$

and

$$\frac{d\omega}{dt} = \mp \frac{\mu}{a} g. \tag{5}$$

Case I. — Suppose the initial angular velocity to be clockwise and

$v_0 < a\omega_0$. Then the sliding at the point of contact is toward the left; therefore $\mathbf{F}$ is directed to the right and consequently positive. Thus

$$\frac{dv}{dt} = \mu g, \tag{4'}$$

$$\frac{d\omega}{dt} = -\frac{\mu g}{a}. \tag{5'}$$

Integrating the last two equations we have

$$v = v_0 + \mu g t. \tag{6}$$

$$\omega = \omega_0 - \frac{\mu g}{a} t. \tag{7}$$

These equations hold until sliding stops, after which the hoop rolls with constant angular and linear velocities. Let t_1 denote the time when sliding stops, that is, when $v = a\omega$. Then

$$v_0 + \mu g t_1 = a\left(\omega_0 - \frac{\mu g t_1}{a}\right),$$

or

$$t_1 = \frac{a\omega_0 - v_0}{2\,\mu g}. \tag{8}$$

Substituting this value of t in equations (6) and (7) we get

$$v_1 = \frac{v_0 + a\omega_0}{2}, \tag{9}$$

and

$$\omega_1 = \frac{v_0 + a\omega_0}{2\,a}, \tag{10}$$

for the linear and the angular velocities of the hoop after the instant when the sliding ceases. The subsequent motion is one of pure rolling with a linear velocity v_1, greater than v_0, and angular velocity ω_1, less than ω_0.

Case II. — Initial rotation clockwise and $v_0 > a\omega_0$. In this case sliding is toward the right, consequently $\mathbf{F}$ is negative and therefore

$$\frac{dv}{dt} = -\ \mu g, \tag{4''}$$

$$\frac{d\omega}{dt} = \frac{\mu g}{a}. \tag{5''}$$

If t_2 denotes the time when sliding stops, in this case, a reasoning similar to the foregoing gives

$$t_2 = \frac{v_0 - a\omega_0}{2\,\mu g}, \tag{8'}$$

$$v_2 = \frac{v_0 + a\omega_0}{2}, \tag{9'}$$

$$\omega_2 = \frac{v_0 + a\omega_0}{2\,a}. \tag{10'}$$

Therefore after the time t_2 the hoop will roll along towards the right with a linear velocity v_2 less than v_0, and with an angular velocity ω greater than ω_0.

Case III. — Suppose the initial rotation to be counter-clockwise. In this case we obtain

$$\frac{dv}{dt} = -\mu g, \tag{4'''}$$

$$\frac{d\omega}{dt} = -\frac{\mu g}{a} \tag{5'''}$$

$$t_3 = \frac{v_0 + a\omega_0}{2\mu g}, \tag{8''}$$

$$v_3 = \frac{v_0 - a\omega_0}{2}, \tag{9''}$$

$$\omega_3 = \frac{v_0 - a\omega_0}{2a}, \tag{10''}$$

where t_3 is the time when sliding ceases.

There are three special cases to be considered:

(a) When $v_0 > a\omega_0$, v_3 is positive, and consequently the hoop goes on rolling towards the right.

(b) When $v_0 = a\omega_0$, $v_3 = 0$, and consequently at $t = t_3$ the hoop comes to rest.

(c) When $v_0 < a\omega_0$, v_3 is negative. Therefore at the instant $t = t_3$ the hoop begins to roll backwards.

PROBLEMS.

1. Discuss the motion of the following bodies rolling down an inclined plane without slipping:

(a) A hollow cylinder of mass m and inner and outer radii r_1 and r_2, respectively.

(b) A hoop of mass m and radius r.

(c) A sphere of mass m and radius r.

(d) A hollow sphere of mass m and inner and outer radii r_1 and r_2, respectively.

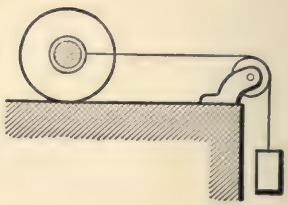
(e) A spherical shell of negligible thickness of mass m and of radius r.

(f) Compare the times of descent in (c) and (e).

2. A sphere is projected, without initial rotation, up a perfectly rough inclined plane. Discuss the motion.

3. A wheel which is rotating about its own axis is placed on a perfectly rough inclined plane. Discuss the motion up the plane.

4. The return trough of a bowling alley is 50 feet long and has a slope of 1 foot in 20 feet. Supposing the contact to be perfectly rough find the time a ball will take to return. The sides of the trough are perpendicular to each other.



5. In the adjoining figure the largest circle represents a solid disk wheel, which rolls along a rough horizontal table under the action of a falling body. The left-hand end of the string is spliced and connected to two smooth rings on the axle of the wheel. The pulley over which the string passes is smooth. Discuss the motion.

6. In the preceding problem suppose the pulley to be rough and to rotate about its axis.

7. Same as problem 5 except that the wheel rolls up an inclined plane.

8. In the preceding problem suppose the pulley to rotate.

9. Same problem as the third illustrative example, p. 231, except that the cylinder is hollow and has a negligible thickness.

10. Same as the preceding problem, but the cylinder rolls up an inclined plane.

11. How can you tell a solid sphere from a hollow one which has exactly the same diameter and mass?

12. Two men of different weights coast down a hill on exactly similar bicycles. Which will reach the bottom of the hill first, the lighter or the heavier man?

13. A thin spherical shell of perfectly smooth inner surface is filled with water and allowed to roll down an inclined plane. Discuss the motion.

14. A hollow cylinder of negligible thickness and perfectly smooth inner surface is filled with water and allowed to roll down an inclined plane. Discuss the motion.

GENERAL PROBLEMS.

1. A sphere of radius a starts from the top of a fixed sphere of radius b and rolls down. If there is no sliding find the position at which they will separate.

2. Two masses m_1 and m_2 are suspended by means of strings which are wound around a wheel and its axle, respectively. The wheel and axle are rigidly connected and are free to rotate about a horizontal axis. Discuss the motion,

(a) When M_1 and M_2, the masses of the wheel and axle, are negligible;

(b) When they are not negligible.

3. In the Atwood machine problem show that if the pulley is not rough enough the acceleration of the two moving masses is $\frac{M - me^{\mu\pi}}{M + me^{\mu\pi}} g$ where μ is the coefficient of friction.

Hint. — If T and T' are the tensile forces in the string on the two sides, $T = T'e^{\mu\pi}$.

4. Same as the third illustrative problem, but the pulley P is supposed to rotate.

5. In the preceding problem suppose the cylinder to roll up an inclined plane.

6. A tape of negligible mass and thickness is wound around the middle of a cylinder. The free end of the tape is attached to a fixed point and then the cylinder is allowed to fall. Show that the cylinder falls with an acceleration of $\frac{2}{3} g$ and the tensile force of the tape is $\frac{1}{3} W$, where W is the weight of the cylinder.

7. In the preceding problem the fixed point is on an inclined plane and the cylinder rolls down the plane.

8. Discuss the motion of a log which moves along its length down an inclined plane, upon two rollers, which stay horizontal.

9. A uniform rod is allowed to fall from a position where its lower end is in contact with a rough plane and it makes an angle α with the horizon. Show that when it becomes horizontal its angular velocity is $\sqrt{\frac{3g}{l} \sin \alpha}$, where l is the length of the rod.

10. Discuss the motion of a cylinder down an inclined plane, supposing the contact to be imperfectly rough, so that the cylinder both slides and rolls.

11. In the preceding problem suppose the cylinder to be hollow.

CHAPTER XII.

IMPULSE AND MOMENTUM.

190. Impulse.—It was stated at the beginning of Chapter VIII that when a force acts upon a body two entirely different mechanical results are produced which are called *work* and *impulse*. The former is the result of the action of force in space. The latter is the result of the action of force in time. We have already discussed work. Impulse is the subject of the present chapter.

191. Measure of Impulse. — If a force which is constant both in direction and magnitude acts upon a particle the impulse which it imparts to the particle equals the product of the force by the time during which it acts. Since time is a scalar while force is a vector, impulse is a vector which has the same direction as the force. If $\mathbf{L}$ denotes the impulse which a constant force $\mathbf{F}$ imparts in the interval of time t, we can write

$$\mathbf{L} = \mathbf{F} \cdot t. \tag{I'}$$

When the force is variable in magnitude or in direction, or in both, we must consider the impulses imparted in infinitesimal intervals of time and add them up. Thus

$$d\mathbf{L} = \mathbf{F}\,dt$$

and

$$\mathbf{L} = \int_0^t \mathbf{F}\,dt. \tag{I}$$

Substituting in the last equation $m\dot{\mathbf{v}}$ for $\mathbf{F}$ we have

$$\begin{aligned} \mathbf{L} &= \int_0^t m\dot{\mathbf{v}}\,dt \\ &= m\int_{v_0}^{v} d\mathbf{v}, \\ &= m\mathbf{v} - m\mathbf{v}_0, \end{aligned} \tag{II}$$

where $\mathbf{v}_0$ and $\mathbf{v}$ are the velocities at the instants $t = 0$ and $t = t$, respectively. If $\mathbf{v}_0$ and $\mathbf{v}$ are parallel, equation (II) may be written in the form

$$L = mv - mv_0. \tag{II$'$}$$

192. Momentum.—The vector magnitude $m\mathbf{v}$ is called *momentum.* Therefore the momentum of a particle equals the product of the mass by the velocity and has the same direction as the latter. Equation (II) states, therefore, that *impulse equals the vector change in momentum.*

PROBLEM.

Show that the component of the impulse along any direction equals the change in the component of the momentum along the same direction, that is,

$$\int_0^t X\,dt = m\dot{x} - m\dot{x}_0, \text{ etc.}$$

193. Dimensions and Units.—Substituting the dimensions of force and time in the definition of impulse and those of mass and velocity in the definition of momentum, we obtain $[MLT^{-1}]$ for the dimensions of both. The C.G.S. unit of both impulse and momentum is the $\dfrac{\text{gm. cm.}}{\text{sec.}}$. The British unit is the pound-second.

Force and Momentum. — Let $\mathbf{F}$ denote the resultant of all the forces acting upon a particle of mass m. Then we have

$$\begin{aligned} \mathbf{F} &= m\dot{\mathbf{v}} \\ &= \frac{d}{dt}(m\mathbf{v}), \end{aligned} \tag{III}$$

which states that the *resultant force experienced by a particle equals the time rate of change of the momentum of the particle.*

In order to extend this result to a system of particles let $\mathbf{F}$ denote the resultant of all the external forces acting upon the system. Further let $\mathbf{F}_i$ be the resultant of all the forces acting upon any one of the particles. Evidently $\mathbf{F}_i$ is the resultant of two sets of forces, namely, those which are

external and those which are internal to the system. Let $\mathbf{F}_i'$ denote the resultant of the external forces acting on the particle, and $\mathbf{F}_i''$ denote the resultant of the internal forces acting upon it, due to its connection with the rest of the system. Then

$$\mathbf{F}_i = \mathbf{F}_i' + \mathbf{F}_i''.$$

But since $\mathbf{F}$ is the resultant of all the external forces acting upon all the particles of the system we have

$$\begin{aligned} \mathbf{F} &= \Sigma \mathbf{F}_i' \\ &= \Sigma (\mathbf{F}_i - \mathbf{F}_i'') \\ &= \Sigma \mathbf{F}_i - \Sigma \mathbf{F}_i''. \end{aligned}$$

The second sum of the left-hand member is the sum of the internal forces and is nil, because the internal forces come in pairs which mutually annul each other. Therefore

$$\begin{aligned} \mathbf{F} &= \Sigma \mathbf{F}_i && \text{(IV}'\text{)} \\ &= \Sigma m\dot{\mathbf{v}} && \text{(IV)} \\ &= \frac{d}{dt}(\Sigma m\mathbf{v}). && \text{(V)} \end{aligned}$$

These are results which are worth noting. Equation (IV) states that *the resultant external force acting upon a system equals and is opposite to the vector sum (or the resultant) of the kinetic reactions of all the particles of the system.*

Equation (V) states that *the resultant external force acting upon a system equals the time rate of change of the resultant momentum of the system.*

PROBLEMS.

(1) Show that the component, along any direction, of the resultant force acting upon a particle equals the rate at which the corresponding component of its momentum changes, that is,

$$X = \frac{d}{dt}(m\dot{x}), \text{ etc.}$$

(2) Show that the component, along any direction, of the resultant external force acting upon a system equals the rate at which the corre-

sponding component of the resultant momentum of the system changes, that is,

$$X = \frac{d}{dt}(\Sigma m\dot{x}), \text{ etc.}$$

194. The Principle of the Conservation of Momentum. — When the resultant external force is zero equation (V) gives

$$\frac{d}{dt}(\Sigma m\mathbf{v}) = 0$$

or

$$\Sigma\,(m\mathbf{v}) = \text{const.} \tag{VI}$$

Therefore *when the sum of the external forces acting upon a system vanishes the resultant momentum of the system remains constant, both in direction and magnitude.* This is the principle of the conservation of momentum. The momenta of the various parts of an isolated system may and, in general, do change, but the vector sum of the momenta of all the particles of the system cannot change either in direction or in magnitude.

PROBLEM.

Show that if the component, along any direction, of the resultant external force vanishes, the corresponding component of the resultant momentum of the system remains constant, that is,

$$\Sigma m\dot{x} = \text{const.}, \text{ when } X = 0.$$

195. Momentum of a System. — The magnitude of the x-component of the resultant momentum of a system may be put in the following forms:

$$\Sigma m\dot{x} = \frac{d}{dt}(\Sigma mx)$$

$$= \frac{d}{dt}(M\bar{x}) \text{ [by equation (I}'\text{), p. 141]}$$

$$\left.\begin{aligned} &= M\dot{\bar{x}}. \\ \Sigma m\dot{y} &= M\dot{\bar{y}}, \\ \Sigma m\dot{z} &= M\dot{\bar{z}}, \end{aligned}\right\} \tag{VII$'$}$$

Similarly

where M is the total mass and $\bar{x}$, $\bar{y}$, and $\bar{z}$ are the coördinates of the center of mass of the system. Combining the last three equations in a single vector equation we obtain

$$\Sigma m\mathbf{v} = M\bar{\mathbf{v}}, \tag{VII}$$

which states that *the resultant momentum of a system equals the product of the total mass of the system by the velocity of its center of mass.*

196. Motion of the Center of Mass of a System. — Combining equations (V) and (VII) we get

$$\mathbf{F} = M\dot{\bar{\mathbf{v}}}, \tag{VIII}$$

which states that the resultant external force acting upon a system equals the product of the total mass of the system by the acceleration of its center of mass. But equation (VIII) is the force equation for a particle of mass M, which is acted upon by a force $\mathbf{F}$. Therefore *the center of mass of a system moves as if the entire mass of the system were concentrated at that point and all the forces acting upon the system were applied to the resulting particle.*

PROBLEM.

Show that when the component, along any direction, of the resultant force acting upon a system vanishes the corresponding component of the velocity of the center of mass remains constant, that is,

$$\dot{\bar{x}} = \text{const., when } X = 0.$$

ILLUSTRATIVE PROBLEM.

A bullet penetrates a fixed plate to a depth d. How far would it penetrate if the plate were free to move in the direction of motion of the bullet?

Let $\mathbf{F}$ be the mean resisting force which the plate offers to the motion of the bullet. When the plate is fixed all the energy of the bullet is expended in doing work against this force. Therefore we have

$$Fd = \tfrac{1}{2} mv^2, \tag{1}$$

where m is the mass and $\mathbf{v}$ the velocity of the bullet. When the target is free to move part of the energy of the bullet is expended in giving the

target and the bullet a common velocity $\mathbf{v}'$. Therefore if d' be the new depth of penetration we have

$$Fd' = \tfrac{1}{2}\, mv^2 - \tfrac{1}{2}\,(m + M)\, v'^2, \tag{2}$$

where M is the mass of the target. Eliminating F between equations (1) and (2) we get

$$d' = \left[1 - \frac{m+M}{m}\left(\frac{v'}{v}\right)^2\right] d. \tag{3}$$

But by the conservation of momentum we have

$$mv = (m + M)\, v'. \tag{4}$$

Therefore eliminating the velocities between equations (3) and (4) we get

$$d' = \frac{M}{M+m}\, d. \tag{5}$$

It is evident from equation (5) that when the target is free, but very large compared with the bullet, the depth penetrated is about the same as when it is fixed.

PROBLEMS.

1. A particle which weighs 2 ounces describes a circle of 1.5 feet radius on a smooth horizontal table. If it makes one complete revolution in every 3 seconds find the magnitude and direction of the impulse imparted by the force, which keeps the particle in the circle,

(a) in one-quarter of a revolution;
(b) in one-half of a revolution;
(c) in three-quarters of a revolution;
(d) in one complete revolution.

2. Find the expression for the impulse imparted to a particle in describing an arc of a circle with uniform speed.

3. Considering the rate of change of the momentum of a particle which describes a uniform circular motion derive the expression for the central force.

4. If we neglect the resistance of the air to the motion of a projectile what can we state with regard to the components of the momentum in the horizontal and vertical directions?

5. A train which weighs 100 tons runs due south at the rate of one mile a minute. Find the lateral force on the western rails due to the rotation of the earth, while the train passes the line of 30° latitude.

6. At what latitude will the force of the preceding problem be a maximum? Determine its amount.

7. Two trains, weighing 150 tons each and moving towards each other at the rate of 40 miles an hour, collide. Find the average force which comes into play if the collision lasts 1.5 seconds.

8. A body explodes while at rest and flies to pieces. If at any instant after the explosion the different parts of the body are suddenly connected, will it move?

9. A shell of mass m explodes at the highest point of its flight and breaks into two parts, the one n times the other. Find the velocity of one piece if the other is brought to rest for an instant by the explosion. The velocity of the shell at the instant of explosion is v.

10. In the preceding problem will the motion of the center of mass of the entire shell be affected by the explosion? Answer this question on the assumption (a) that there is no air resistance, (b) that there is an air resistance.

11. A man walks from one end to the other of a plank placed on a smooth horizontal plane. Show that the plank is displaced a distance

$$\frac{M}{M+m}l,$$

where M and m are the masses of the man and of the plank, respectively, and l is the length of the plank.

12. A shell, which weighs 150 pounds, strikes an armor plate with a velocity of 2000 feet per second and emerges on the other side with a velocity of 500 feet per second. Supposing the resisting force to be uniform, find its magnitude and show that the impulse produced by it equals the change in the momentum of the shell while plowing through the plate. The plate is 10 inches thick.

COLLISION AND IMPACT.

197. Central Collision. — If two bodies collide while moving along the line which joins their centers of mass the collision is said to be *central.* In order to fix our ideas suppose the colliding bodies to be spheres, then Fig. 118 represents roughly the state of affairs during the collision. For a short interval of time after the spheres come into contact their centers approach each other and a little deformation takes place in the neighborhood of the point of contact at the end of which the centers of the spheres are, just for an instant, at rest with respect to one another, and are moving with a

common velocity. Then the deformed parts of the spheres begin to regain, at least partially, their original forms and cause the spheres to separate.

The process of collision may, therefore, be divided into two parts. The first part lasts from the initial contact at $t = 0$ until the instant when the centers of the spheres are nearest together at $t = t_1$. The second part begins at $t = t_1$ and lasts until the spheres separate at $t = t_1'$. The impulse imparted to each body during the first part of the collision is called the *impulse of compression*, while that imparted during the second part is called the *impulse of restitution*.

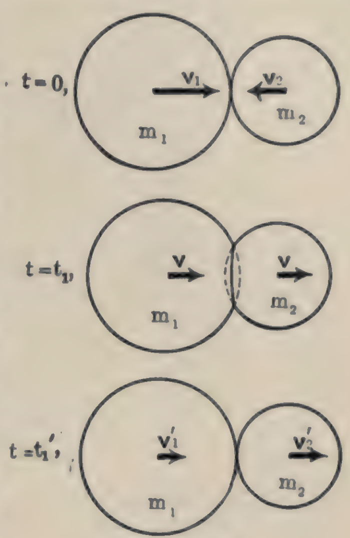


FIG. 118.

Let m_1 and m_2 be the masses of the colliding bodies, $\mathbf{v}_1$ and $\mathbf{v}_2$ be their velocities just before and $\mathbf{v}_1'$ and $\mathbf{v}_2'$ just after the collision, and let $\mathbf{v}$ be their common velocity at the instant of *maximum compression*, that is, when the distance between the centers of mass is shortest. Further, let $\mathbf{L}$ and $\mathbf{L}'$ denote the impulses of compression and of restitution, respectively. Then we have

$$L = \int_0^{t_1} F\,dt = m_1\,(v - v_1) = -\,m_2\,(v - v_2),$$

$$L' = \int_{t_1}^{t_1'} F\,dt = m_1\,(v_1' - v) = -m_2\,(v_2' - v).$$

The foregoing relations follow at once from the definition of impulse and from the fact that the colliding bodies form a system which is not acted upon by external forces, and consequently the sum of their momenta remains constant during the collision.

198. Coefficient of Restitution. — It is found by experiment that the ratio of the impulse of restitution to the impulse of compression depends only upon the nature of the bodies in collision. The ratio, therefore, is a constant of the substances in collision. This constant is called the *coefficient of restitution,* and is generally denoted by the letter e. Thus

$$\begin{aligned} e &= -\frac{L'}{L} \qquad \text{(IX)} \\ &= -\frac{v_1' - v}{v - v_1} \\ &= -\frac{v_2' - v}{v - v_2}. \end{aligned}$$

Eliminating v we obtain

$$e = -\frac{v_1' - v_2'}{v_1 - v_2}. \qquad \text{(X')}$$

But $(v_1 - v_2)$ and $(-v_1' + v_2')$ are the velocities of the first body relative to the second, just before and just after the collision. Denoting them by V and V', respectively, we obtain

$$\begin{aligned} e &= \frac{V'}{V} \qquad \text{(X)} \\ &= \frac{\text{relative velocity after impact}}{\text{relative velocity before impact}}. \end{aligned}$$

199. Resiliency. — When two bodies rebound after collision they are said to have *resiliency,* and the contact is called *elastic contact.* The coefficient of restitution is a measure of the resiliency of the colliding bodies. When $e = 1$ the resiliency of the colliding bodies is perfect and the contact is said to be *perfectly elastic.*

The coefficient of restitution cannot have a value greater than unity, as will be seen from a consideration of the transformation of energy which takes place during collision. At the beginning of the collision the bodies have a certain amount of kinetic energy which depends upon their relative

velocity at that instant. During the compression part of the collision a fraction of their energy is transformed into potential energy of compression and the rest into heat energy. During the restitution a fraction of the potential energy is transformed into kinetic energy and the rest into heat energy. Thus, in general, the kinetic energy at the end of the collision is less than that at the beginning. Therefore the relative velocity at the end of the collision is less than that at the beginning. Thus the coefficient of restitution is, in general, less than unity. If none of the energy, which is due to the relative motion of the colliding bodies, is lost in the form of heat, it is all transformed into potential energy during the compression and back into kinetic energy during the restitution. In this case the relative velocity at the end of the collision equals that at the beginning, which makes the coefficient of restitution unity.* The relative velocity at the end of the collision may be made greater than that at the beginning by having explosives at the point of contact. But this does not come in the definition of the coefficient of restitution. Therefore unity is the highest value of e. When all the kinetic energy is transformed into heat during the collision the bodies have no relative velocity after the collision. In this case the contact is called *perfectly inelastic*. Evidently e is zero when the contact is perfectly inelastic. Therefore the value of e lies between zero and unity. The values of the coefficient of restitution are 0.95 for glass on glass, 0.81 for ivory on ivory, and 0.15 for lead on lead.

200. Loss of Kinetic Energy of Colliding Bodies. — The kinetic energy of a system equals the kinetic energy due to the linear motion of the system with the velocity of its

* In working out problems in which the contact is perfectly elastic instead of introducing the coefficient of restitution make use of the principle of the conservation of energy. The conservation of dynamical energy holds only when the contact is perfectly elastic. But the conservation of momentum and the conservation (general) of energy are true under all circumstances.

center of mass plus the kinetic energy of its parts due to their motion relative to the center of mass. Collision does not affect the motion of the center of mass of the system formed of the colliding bodies, because the forces which arise during the collision are internal forces. Therefore that part of its kinetic energy which is due to the motion of its center of mass does not suffer any loss. The loss occurs in that part of the energy which is due to the motion of the parts of the system with respect to the center of mass. Referring all the velocities to the center of mass and denoting the loss of kinetic energy by T_l, we have

$$T_l = (\tfrac{1}{2}\, m_1 v_1{}^2 + \tfrac{1}{2}\, m_2 v_2{}^2) - (\tfrac{1}{2}\, m_1 v_1'^2 + \tfrac{1}{2}\, m_2 v_2'^2),$$

where v_1 and v_2 are the velocities just before and v_1' and v_2' the velocities just after the collision.

We can eliminate v_1' and v_2' from this expression for T_l by means of the principle of the conservation of momentum and the definition of e. According to the former

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

and by (X′)

$$v_1' - v_2' = -e\,(v_1 - v_2).$$

Eliminating v_2' between the last two equations we have

$$v_1 - v_1' = \frac{m_2}{m_1 + m_2}(v_1 - v_2)(1 + e).$$

The following changes in the expression of T_l are effected by means of the last three equations.

$$\begin{aligned}
T_l &= \tfrac{1}{2}\, m_1 (v_1{}^2 - v_1'^2) + \tfrac{1}{2}\, m_2\, (v_2{}^2 - v_2'^2) \\
&= \tfrac{1}{2}\, m_1\, (v_1 - v_1')(v_1 + v_1') + \tfrac{1}{2}\, m_2\, (v_2 - v_2')(v_2 + v_2') \\
&= \tfrac{1}{2}\, m_1\, (v_1 - v_1')(v_1 - v_2 + v_1' - v_2') \\
&= \tfrac{1}{2}\, m_1\, (v_1 - v_1')(v_1 - v_2)(1 - e) \\
&= \tfrac{1}{2}\, \frac{m_1 m_2}{m_1 + m_2}\, (v_1 - v_2)^2\, (1 - e^2). \qquad \text{(XI)}
\end{aligned}$$

When the colliding bodies are perfectly elastic then $e = 1$

and $T_l = 0$; on the other hand if the bodies are perfectly inelastic, $e = 0$; therefore, $T_l = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}(v_1 - v_2)^2$.

201. Impact.—When the mass of one of the colliding bodies is very large compared with that of the other the velocity of the former with respect to the center of mass of the colliding system does not change appreciably during the collision. In such a case the body with the greater mass is considered to be fixed and the collision is called an *impact.* The impact of a falling body when it strikes the ground is a case in point.

The velocities of the larger mass before and after the collision, as well as the common velocity at the instant of maximum compression, are negligible. Therefore making these changes in the expressions for L, L', e, and T_l and dropping the subscripts we obtain

$$L = mv,$$

$$L' = -mv,$$

$$e = \frac{v'}{v}, \qquad (X'')$$

and

$$T_l = \tfrac{1}{2} mv^2 (1 - e^2), \qquad (XI')$$

where m is the mass of the impinging body, while v and v' are its velocities just before and just after impact, respectively.

ILLUSTRATIVE EXAMPLE.

A ball which is thrown vertically down from a height h rises to the point of projection after impinging against a horizontal floor. Find the velocity of projection and the loss in energy.

Let v_0 be the velocity of projection, then the velocities just before and just after the impact are

$$v = \sqrt{v_0^2 + 2gh} \quad \text{and} \quad v' = \sqrt{2gh},$$

respectively. But $v' = ev$. Therefore

$$v_0^2 = \frac{1 - e^2}{e^2} 2gh,$$

$$T_l = \tfrac{1}{2} mv^2 (1 - e^2) = \tfrac{1}{2} mv_0^2.$$

Discussion. — The energy lost during the impact equals the kinetic energy of projection, as would be expected from the conservation of energy.

When $e = 1$, $v_0 = 0$ and $T_l = 0$. In other words when the ball is perfectly elastic it will rise to the height from which it is dropped. The entire kinetic energy is transformed, during the impact, into potential energy and back to kinetic energy without any loss.

When $e = 0$, $v_0 = \infty$ and $T_l = \infty$, that is, if the contact is perfectly inelastic no value of the velocity of projection will enable the ball to rebound after the impact.

PROBLEMS.

1. Show that when two perfectly elastic spheres of equal mass collide centrally they exchange velocities.

2. A ball of mass m_1, impinging directly on another ball of mass m_2 at rest, comes to rest. Show that $m_1 = em_2$.

3. Two perfectly elastic balls collide directly with equal velocities. The relation between their masses is such that one of them is reduced to rest. Find the relation.

4. A ball which is dropped on a horizontal floor from a height h reaches a height equal to $\frac{1}{3} h$ at the second rebound. Find the coefficient of restitution.

5. A metal patched bullet strikes a wall normally with a velocity of $1200 \frac{\text{ft.}}{\text{sec.}}$. With what velocity will it rebound if $e = 0.4$?

6. Show that if two equal balls collide centrally with velocities $\frac{1+e}{1-e} v$ and $-v$, the one which has the former velocity will come to rest.

7. A bullet strikes a vertical target normally and rebounds. Find the relation between the distances of the foot of the target from the rifle and from the place where the bullet strikes the ground.

8. Two perfectly elastic equal balls collide with velocities inversely as their masses. Find the velocities after collision.

9. Two billiard balls collide centrally with velocities of 8 feet per second and 16 feet per second. Supposing $e = 0.8$, find the final velocities.

10. A ball is dropped from the top of a tower, at the same instant that another ball of equal mass is projected upward from the base of the tower, with a velocity just enough to raise it to the top of the tower. Show that if the balls collide centrally the falling ball will rise, on the rebound, to a height $\frac{h}{4}(3 + e^2)$ above the ground, where h is the height of the tower.

11. Two spheres of masses m and $2\,m$ moving with equal velocities along two lines at right angles to each other collide at the instant when their centers are on the line of motion of the smaller sphere. Show that if the contact is smooth and $e = 0.5$ the smaller sphere will come to rest, and find the direction and magnitude of the velocity of the larger sphere.

12. In the preceding problem let $m = 500\ gm$, $v = 40$ cm. per second and find

(a) the impulse, its magnitude and direction;

(b) the loss of energy.

13. A metal patched bullet which weighs 1.5 ounces strikes a rock, normally, with a velocity of 1500 feet per second. Find the velocity with which it will rebound and the impulse given to the rock; $e = 0.5$.

14. A body impinges against another body which has a mass n times as large. Show that if the larger body is at rest and the contact inelastic the loss of energy is $\frac{n}{n+1}$ times its value before the collision.

15. A particle is projected up a smooth inclined plane with a velocity $\sqrt{gh}$; simultaneously a particle of equal mass is allowed to slide down the inclined plane. The two collide somewhere on the plane. Find the velocities with which the particles will arrive at the bottom of the plane. h = the height of the inclined plane.

16. Two small spheres of masses m and $2\,m$ move in a smooth circular groove on a horizontal table with equal speeds in opposite directions. Find the position of the second collision relative to the first; $e = 0.6$.

17. In the preceding problem find the interval of time between the first and the eleventh collision, under the following assumptions — the radii of the particles are negligible compared with that of the circular groove, which equals 50 cm., the common speed of the spheres just before the first collision is 500 cm. per second, the time of collision is negligible.

202. Efficiency of a Blow. — A blow may be struck to produce one or the other of two distinct results. The object of a blow from a hammer in driving a nail is quite different from that of a blow in shaping a rivet. Efficiency in the first case means greatest amount of driving with the least amount of deformation, while in the second case it means greatest amount of smashing with the least amount of driving. Therefore the efficiency of a blow is different for these two cases. We may define

$$\text{Driving efficiency} = \frac{\text{energy expended in driving}}{\text{total energy expended}}.$$

$$\text{Smashing efficiency} = \frac{\text{energy expended in deforming}}{\text{total energy expended}}.$$

Consider the case of a blow which drives a nail or a pile. Let M be the mass of the hammer, m the mass of the pile, v the velocity of the hammer just before impact, v' the velocity just after impact. The contact between the hammer and the pile may be regarded as inelastic, therefore just after the impact both the pile and the hammer have the same velocity v'. In other words, immediately after the impact there is an amount of energy equal to $\frac{1}{2}(M+m)v'^2$ available for driving the pile, while the balance of the energy of the hammer, that is, $\frac{1}{2}Mv^2 - \frac{1}{2}(M+m)v'^2$, is expended during the impact in producing permanent deformation, heat, and sound. Substituting these in the two definitions for the efficiency of a blow we obtain

$$\text{Driving efficiency} = \frac{(M+m)v'^2}{Mv^2}.$$

$$\text{Smashing efficiency} = 1 - \frac{(M+m)v'^2}{Mv^2}.$$

Immediately after the impact practically all the momentum of the hammer relative to the earth will be in the hammer and the pile; therefore we can write

$$Mv = (M+m)v'.$$

Eliminating the velocities between the last equation and the above expressions for the two efficiencies we obtain

$$\left.\begin{aligned} \text{Driving efficiency} &= \frac{M}{M+m}. \\ \text{Smashing efficiency} &= \frac{m}{M+m}. \end{aligned}\right\} \qquad \text{(XII)}$$

It is evident from these expressions that for driving piles or nails the ram or the hammer head must have a large mass

compared with the pile or the nail. On the other hand, for shaping rivets the anvil must have a large mass compared with the hammer.

203. Motion where Moving Mass Varies. — If the moving mass varies, as in the case of an avalanche, the relation between impulse and momentum still holds, that is, impulse equals the change in the momentum. We have to take into account, however, the change in the momentum of the mass which is continually added to the moving system as well as that of the original mass. Let a mass dm be added in the time dt; then supposing dm to have been initially at rest the total change in the momentum is $m\,d\mathbf{v} + \mathbf{v}\,dm$, where the first term is the increase in the momentum of m and the second term is the increase in the momentum of dm. Therefore the impulse given by the resultant force $d\mathbf{F}$ in the time dt is

$$\mathbf{F}\,dt = m\,d\mathbf{v} + \mathbf{v}\,dm,$$

or

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} + \mathbf{v}\frac{dm}{dt}$$

$$= \frac{d}{dt}(m\mathbf{v}),$$

which is the same equation as (III), except that in (III) m was considered to be constant, while here it is considered as a variable.

If dm has an initial velocity $\mathbf{u}$, then the change in the momentum of dm is $(\mathbf{v} - \mathbf{u})\,dm$. Therefore

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} + (\mathbf{v} - \mathbf{u})\frac{dm}{dt}$$

$$= \frac{d}{dt}(m\mathbf{v}) - \mathbf{u}\frac{dm}{dt}. \qquad \text{(XIII)}$$

ILLUSTRATIVE EXAMPLES.

1. A jet of water strikes a concave vessel with a velocity of 80 feet per second and then leaves it with a velocity which has the same magnitude as the velocity of impact but makes an angle of 120° with it. If the diameter of the jet is 1 inch find the force necessary to hold the concave vessel in position.

The force experienced by the vessel equals the rate at which it receives momentum. Suppose the vessel to be symmetrical with respect to the axis of the jet, as in Fig. 119, then by symmetry there can be no resultant force on the vessel in a direction perpendicular to the axis of the jet. Therefore we need to consider only the change in momentum along the axis. Let m be the mass of water delivered by the jet in the time t, v the velocity of impact, and α the change in the direction of flow. Then the force is

$$\begin{aligned} F &= \frac{mv - mv\cos\alpha}{t} \\ &= \frac{m}{t} v(1 - \cos\alpha) \\ &= \frac{W}{gt} v(1 - \cos\alpha) \\ &= \frac{w_1 A}{g}\frac{l}{t} v(1 - \cos\alpha) \\ &= \frac{w_1 A v^2}{g}(1 - \cos\alpha), \end{aligned}$$

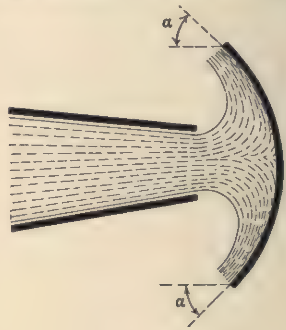


FIG. 119.

where A is the area of the cross-section of the jet and w_1 is the weight of a cubic foot of water. Replacing the various magnitudes by their numerical values we obtain

$$F = \frac{62.5\,\frac{\text{lb.}}{\text{ft.}^3} \times \pi\left(\frac{1}{24}\text{ ft.}\right)^2 \times \left(80\,\frac{\text{ft.}}{\text{sec.}}\right)^2 \times (1 + \cos 60^\circ)}{32\,\frac{\text{ft.}}{\text{sec.}^2}}$$

$$\doteq 102.3 \text{ lb.}$$

DISCUSSION. — It is evident from the general expression of F that its value depends upon α and varies between zero for $\alpha = 0$ and $\frac{2\,w_1 A v^2}{g}$ for $\alpha = \pi$. When $\alpha = \frac{\pi}{2}$, $F = \frac{w_1 A v^2}{g}$.

2. A uniform chain is hung from its upper end so that its lower end just touches an inelastic horizontal table, and then it is allowed to fall. Find the force which the table will experience at any instant during the fall of the chain.

The force is partly due to the weight of that part of the chain which is on the table at the instant considered and partly due to the rate at which the table is receiving momentum. Let x be the height of the upper end of the chain above the table, l the total length, and ρ the mass per unit length. Then $\rho g\,(l - x)$ is the weight of that part of the chain which is on the table. On the other hand the momentum which the table receives in the interval of time dt is $\rho v\, dt \cdot v$. Therefore the rate at which it receives momentum is ρv^2, where v is the velocity of that part of the chain which is above the ground. This velocity is the same as that of the upper end of the chain, therefore

$$v = \sqrt{2\,g\,(l - x)}.$$

Hence the total force is

$$\begin{aligned} F &= \rho\,(l - x)\,g + \rho \cdot 2\,g\,(l - x) \\ &= 3\,\rho\,(l - x)\,g. \end{aligned}$$

Discussion. — When $x = l$, that is, at the beginning of the motion, the force is zero. When $x = 0$, that is, at the end of the motion, it is $3\,\rho l g$, or three times the weight of the chain. As soon as the entire chain comes to rest on the table the force equals the weight of the chain.

3. A spherical raindrop, descending by virtue of its weight, receives continuously, by precipitation of vapor, an accession of mass proportional to the surface. Find the velocity at any instant.

The external force acting upon the drop at any instant equals the rate at which its momentum changes, therefore

$$mg = \frac{d}{dt}\,(mv), \tag{1}$$

where m, the mass of the drop, is variable. Since the accession of mass is proportional to the surface the rate of change of radius of the drop will be constant. Let a be the radius of the drop when it begins to fall, r its radius at any later instant, and k the rate at which r increases. Then at any instant

$$\begin{aligned} m &= \tau\,\tfrac{4}{3}\,\pi r^3 \\ &= \tau\,\tfrac{4}{3}\,\pi\,(a + kt)^3, \end{aligned}$$

where τ is the density of water. Substituting this expression for m in equation (1)

$$g\,(a+kt)^3 = \frac{d}{dt}\,[(a+kt)^3\,v]$$

$$= (a+kt)^3\frac{dv}{dt} + 3\,(a+kt)^2\,kv$$

and
$$\frac{dv}{dt} + \frac{3\,k}{a+kt}\,v = g,^* \tag{2}$$

the integral of which is $v = e^{-\int\frac{3\,k}{a+kt}\,dt}\left[\int g e^{\int\frac{3\,k}{a+kt}dt}\,dt + c\right]\cdot$

$$\therefore \quad v = e^{-3\log(a+kt)}\left[\int g e^{3\log(a+kt)}\,dt + c\right]$$

$$= (a+kt)^{-3}\left[g\int(a+kt)^3\,dt + c\right]$$

$$= (a+kt)^{-3}\left[g\left(a^3t + \frac{3\,a^2kt^2}{2} + \frac{3\,ak^2t^3}{3} + \frac{k^3t^4}{4}\right) + c\right]$$

$$= (a+kt)^{-3}\left[\frac{g}{4}\,(4\,a^3t + 6\,a^2kt^2 + 4\,ak^2t^3 + k^3t^4) + c\right].$$

Let $v = 0$ when $t = 0$; then $c = 0$.

$$\therefore \quad v = \frac{gt}{4}\,\frac{4\,a^3 + 6\,a^2kt + 4\,ak^2t^2 + k^3t^3}{(a+kt)^3}$$

$$= \frac{gt}{4}\left(1 + \frac{a}{r} + \frac{a^2}{r^2} + \frac{a^3}{r^3}\right)\cdot$$

PROBLEMS.

1. Find the pressure upon the canvas roof of a tent produced by a shower. The following data are given — the raindrops have a velocity of $50\,\frac{\text{ft.}}{\text{sec.}}$ at right angles to the roof; the intensity of the shower is such as to produce a deposit of 0.2 inch per hour; 1 cubic foot of water weighs 62.5 pounds.

2. Find the pressure on horizontal ground due to the impact of a column of water which falls vertically from a height of 500 feet.

3. Water flowing through a pipe at the rate of $100\,\frac{\text{cm.}}{\text{sec.}}$ is brought to

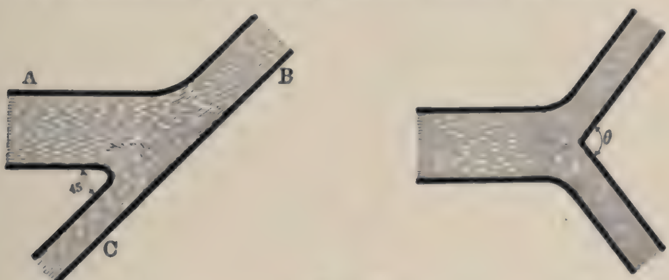
* Equation (2) is of the form $\frac{dy}{dx} + Py = Q$, which is the typical linear equation, with the integral $y = e^{-\int P\,dx}\left[\int Qe^{\int P\,dx}\,dx + c\right]$.

rest in 0.1 second by closing a valve at the lower end. Find the increase of pressure produced near the valve in both the C.G.S. and the British units. The length of the pipe is 500 meters.

4. A jet of water strikes a blade of a turbine normally. If the velocity of the jet is 150 feet per second, find the pressure it exerts on the blade, (a) when the blade is fixed; (b) when it has a velocity of 50 feet per second along the jet.

5. Figure 120a represents a horizontal trough with smooth vertical walls. The stream is supposed to have the same speed, 6 miles per hour, in all three parts of the trough. The stream in C is one-third of that in B. Find the force on the wall BC. The cross-section of the stream in A is 5 feet by 3 feet.

6. In the preceding problem suppose the branch C to be closed.



(a) FIG. 120. (b)

7. A stream of water flowing in a horizontal direction is divided into two equal streams, as shown in Fig. 120b. Supposing the velocity of the water to remain unchanged derive an expression for the force on the obstacle, and discuss it for special values of θ.

8. In the preceding problem suppose the velocity of the stream to be 5 miles per hour, its cross-section before it is divided to be 4 feet by 2 feet, and $\theta = 120°$.

9. In the preceding problem take $\theta = \pi$.

10. In (8) take $\theta = \frac{3\pi}{2}$.

11. In (8) take $\theta = 2\pi$.

12. A machine gun delivers 500 bullets per minute with a velocity of 1800 feet per second. If the bullets weigh 0.5 ounce each find the average force on the carriage of the gun.

13. A train scoops up 1500 pounds of water into the tender from a trough 500 yards long while making 50 miles per hour. Find the added resistance to the motion of the train.

204. Oblique Impact of a Particle upon a Fixed Plane. Case I. Smooth Contact. — Let v_t and v_n, Fig. 121, be the components of the velocity along the plane and along the normal, respectively, just before the impact; and let v_t' and v_n' be the corresponding components just after the impact. Since the plane is smooth, no horizontal forces arise during the impact; hence the horizontal component of the momentum remains constant. Therefore

$$mv_t = mv_t',$$

or

$$v_t = v_t'.$$

So far as the vertical component is concerned the impact is direct; therefore

$$v_n' = ev_n.$$

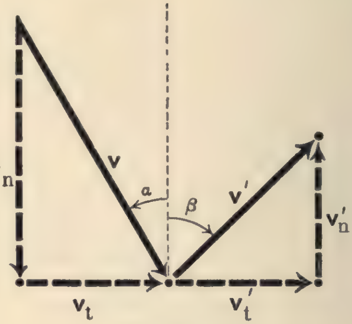


FIG. 121.

Denoting by α and β the angles which the resultant velocity makes with the normal just before and just after the impact we obtain

$$\tan\alpha = \frac{v_t}{v_n},$$

$$\tan\beta = \frac{v_t}{ev_n}.$$

$$\therefore \quad \tan\alpha = e\tan\beta. \qquad \text{(XIV)}$$

DISCUSSION. — When the contact is perfectly elastic $e = 1$; therefore the angle of incidence equals the angle of reflection as in the case of the reflection of light. In this case the magnitude of the velocity is not changed by the impact, as is to be expected from the conservation of energy. When the contact is imperfectly elastic the angle of reflection lies between $\frac{\pi}{2}$ and the angle of incidence, while the normal component of the velocity and consequently the magnitude of the total velocity is diminished. When the contact is perfectly inelastic $e = 0$, and since α is not zero β must be $\frac{\pi}{2}$ in order that $e\tan\beta$ may have a finite value. Therefore in this case the particle slides along the plane after the collision.

205. Case II. Rough Contact. — When the plane is rough frictional forces come into play and change the tangential component of the momentum. Let **F** be the tangential force due to friction, **N** the normal force, and μ the coefficient of friction; then we have

$$L_n = \int_0^T N\,dt = -mv_n,$$

$$L_n' = \int_T^{T'} N\,dt = mv_n';$$

$$\therefore \quad e = \frac{-L_n'}{L_n} = \frac{v_n'}{v_n},$$

$$L_t = \int_0^T F\,dt = \int_0^T \mu N\,dt = -\mu m v_n,$$

$$L_t' = \int_T^{T'} F\,dt = \int_T^{T'} \mu N\,dt = -e\mu m v_n.$$

But $$L_t + L_t' = mv_t' - mv_t.$$

Therefore $$m(v_t' - v_t) = -m\mu(1+e)v_n$$

and $$v_t' = v_t - \mu(1+e)v_n.$$

Substituting this value of v_t' in the expression for $\tan\beta$, which is obtained from Fig. 121, we get

$$\tan\beta = \frac{v_t'}{v_n'} = \frac{v_t - \mu(1+e)v_n}{ev_n}.$$

Eliminating v_t between the last equation and the relation $\tan\alpha = \frac{v_t}{v_n}$ we obtain

$$e\tan\beta = \tan\alpha - \mu(1+e). \tag{XV}$$

Discussion. — When $\mu = 0$, equation (XV) reduces to equation (XIV). When $\mu = \infty$, $\tan\beta = -\infty$ or $\beta = -\frac{\pi}{2}$; therefore the particle slides along the plane towards the left. When $e = 0$ and $\tan\alpha > \mu$, $\tan\beta = \infty$ and $\beta = \frac{\pi}{2}$; therefore the particle slides along the plane towards the right.

When $e = 0$ and $\tan \alpha < \mu$, $\tan \beta = -\infty$ and $\beta = -\frac{\pi}{2}$; therefore the particle is reflected towards the left and slides along the plane.

PROBLEMS.

1. A perfectly elastic ball impinges obliquely on another ball at rest. Prove that their masses are equal if, after impact, the balls move at right angles.

2. A billiard ball strikes simultaneously two billiard balls at rest, and comes to rest. Show that the coefficient of restitution is $\frac{2}{3}$.

3. A particle slides down a smooth inclined plane and then rebounds from a horizontal plane. Find the range of the first rebound.

4. A bullet strikes a target at 45° and rebounds at the same angle. Prove that $e = \frac{1 - \mu}{1 + \mu}$, where μ is the coefficient of friction.

5. Four smooth rods, which form a square, are fixed on a smooth horizontal plane. A particle which is projected from one corner of the square strikes an adjacent corner after three reflections; show that

$$\tan \alpha = \frac{e\,(1 + e)}{1 + e\,(1 + e)},$$

where α is the angle the initial velocity makes with the rod joining the two corners and e is the coefficient of restitution.

6. In the preceding problem discuss the values of α for special values of e.

7. Derive an expression for the percentage of energy lost during oblique impact (a) when the contact is smooth; (b) when the contact is rough.

8. Two billiard balls which are in contact are struck, simultaneously, by a third ball moving with a velocity v, in a direction perpendicular to the line of centers of the first two. Supposing the table to be perfectly smooth find the velocity of each ball after impact.

9. In the preceding problem obtain the expression for the loss of energy and find its value for the following special cases. The balls weigh 6 ounces each.

(a) $v = 16$ feet per second, $e = 0.8$.

(b) $v = 20$ feet per second, $e = 0.5$.

10. A ball impinges against another ball which has twice as large a mass and is at rest. The smaller ball has a velocity of 60 feet per second in a direction which makes 135° with the line of centers. Find the velocities after impact; $e = 0.5$.

GENERAL PROBLEMS.

1. A gun is free to move on smooth horizontal tracks. Show that the loss of energy due to recoil is $\frac{m}{M+m}E$, where M and m are the masses of the gun and the projectile respectively, and E is the kinetic energy which is transmitted to the gun and projectile.

2. In the preceding problem compare the velocities of the projectile when the gun is fixed and when free to move. Also show that the actual angle α' at which the projectile leaves the gun is given by $\tan \alpha' = \frac{M+m}{M}$ $\tan \alpha$, where α is the angle which the gun makes with the horizon.

3. A man stands on a plank of mass m, which is on a perfectly smooth horizontal plane. He jumps upon another plank of the same mass, then back upon the first plank. Find the ratio of the velocities of the two planks if the mass of the man is M.

4. A stream of water delivering 1000 gallons per minute, at a velocity of $20\ \frac{\text{ft.}}{\text{sec.}}$, strikes a plane (1) normally, (2) at an angle of 30°. Find the force exerted on the plane.

5. A uniform chain is held coiled up close to the edge of a smooth table, with one end hanging over the edge. Discuss the motion of the chain when it is allowed to fall, supposing the part hanging over the edge to be very small at the start of the motion.

6. In the preceding problem show that the acceleration is constant if the density of the chain varies as the distance from that end of the chain which is in motion.

7. A mass of snow begins to slide down a regular slope, accumulating more snow as it moves along, thus forming an avalanche. Supposing the path cleared to be of uniform depth and width, show that the acceleration of the avalanche is constant.

8. A ball falls on a floor from a height h and rebounds each time vertically.

(a) Show that
$$T = \frac{1+e}{1-e}\sqrt{\frac{2h}{g}},$$
where T is the total time taken by the ball to come to rest. Find the value of T for $h = 25$ feet and $e = 0.5$.

(b) Show that
$$H = \frac{1+e^2}{1-e^2}h,$$
where H is the total distance described. Find the value of H for $h = 25$ feet and $e = 0.5$.

9. A shell explodes at the highest point of its path and breaks up into two parts, the centers of mass of which lie in the line of motion. Find the velocity of the pieces just after explosion, taking m for the mass of the shell, n for the ratio of the masses of the pieces, v for the velocity of the shell just before explosion, and E for the energy imparted to the pieces by the explosion.

10. A particle slides down a smooth inclined plane which is itself free to move on a smooth horizontal plane. Discuss the motions of the particle and of the plane.

11. After falling freely through a height h a particle of mass m begins to pull up a greater mass M, by means of a string which passes over a smooth pulley. Find the distance through which it will lift M.

12. A smooth inclined plane which is free to move on a smooth horizontal plane is so moved that a particle placed on the inclined plane remains at rest. Discuss the motion of the plane.

13. A disk and a hoop slide along a smooth horizontal plane with the same velocity v, then begin to roll up the same rough inclined plane. How high will each rise?

14. A ball, moving with a velocity v, collides directly with a ball at rest. The second ball in its turn collides with a third ball at rest. If the masses of the first and last ball are m_1 and m_3, respectively, show that the velocity acquired by the third ball is greatest when the mass of the second ball satisfies the relation $m_2 = \sqrt{m_1 m_3}$.

15. Find the maximum velocity acquired by the third ball of the preceding problem.

16. A billiard ball, moving at right angles to a cushion, impinges directly on an equal ball at rest at a distance d from the cushion. Show that they will meet again at a distance $\frac{2e^2}{1+e}d$ from the cushion.

17. A ball is projected from the middle point of one side of a billiard table, so that it strikes an adjacent side first, then the middle of the opposite side. Show that if l is the length of the adjacent side, the ball strikes the adjacent side at a point $\frac{l}{1+e}$ from the corner it makes with the opposite side.

18. A simple pendulum hanging vertically has its bob in contact with a vertical wall. The bob is pulled away from the wall and then it is let go. If e is the coefficient of restitution find the time it will take the pendulum to come to rest.

19. A particle strikes a smooth horizontal plane with a velocity v,

making an angle α with the plane, and rebounds time after time. Prove that

$$\text{(a)}\quad T = \frac{2\,v \sin \alpha}{g\,(1 - e)}, \qquad\qquad \text{(b)}\quad R = \frac{v^2 \sin^2 \alpha}{g\,(1 - e)},$$

where T is the total time of flight after the first impact, R the total range, and e the coefficient of restitution.

20. In the preceding problem find the values of T and R for the following special cases:

(a) $v = 500$ meters per second, $\alpha = 30°$, $e = 0.5$.

(b) $v = 500$ meters per second, $\alpha = 90°$, $e = 0.9$.

21. A particle is projected horizontally from the top of a smooth inclined plane. Derive an expression for the time at the end of which the particle stops rebounding and slides down the plane. Compute its value for the following special cases:

(a) $v_0 = 500$ feet per second, $\alpha = 45°$, $e = 0.5$.

(b) $v_0 = 500$ feet per second, $\alpha = 30°$, $e = 0.3$.

22. In the preceding problem find the distance the particle moves along the plane before it stops rebounding.

23. In problem 21 find the velocity of the particle at the instant it stops rebounding.

24. A bead slides down a smooth circular wire, which is in a vertical plane, and strikes a similar bead at the lowest point of the wire. If during the collision the first bead comes to rest, show that the second bead will rise to a height e^2h and on its return will follow the first bead to a height $e^2\,(1 - e)^2\,h$, where h is the height from which the first bead falls.

25. Two equal spheres, which are in contact, move in a direction perpendicular to their line of centers and impinge simultaneously on a third equal sphere which is at rest. Supposing the contacts to be perfectly smooth and elastic find the velocity of each sphere after the collision.

26. A bullet hits and instantly kills a bird, while passing the highest point of its trajectory. Supposing the bullet to stay imbedded in the bird, and the bird to have been at rest when shot, find the distance between the place of firing and the point where the bird strikes the ground.

27. Two particles of masses m_1 and m_2 are connected by an inextensible string of negligible mass. The second particle is placed on a smooth horizontal table while the first is allowed to fall from the edge of the table. When the falling particle reaches a distance h from the top of the table the string becomes tight. Find the velocity with which the second particle begins to move.

28. A uniform chain lies in a heap close to the edge of a horizontal table. One end of the chain is displaced from the edge of the table so that it begins to fall. Show that when the last portion of the chain leaves the table the chain will have a velocity of $\sqrt{\frac{2\,gl}{3}}$, where l is the length of the chain.

29. A uniform plank is placed along the steepest slope of a smooth inclined plane. Show that if a man runs down the plank making its length in the time given by

$$t^2 = \frac{2\,M}{M + m} \cdot \frac{a}{g \sin \alpha}$$

the plank remains stationary during his motion.

30. A number of coins of equal mass are placed in a row on a smooth horizontal plane, each coin being in contact with its two neighbors. A similar coin is projected along the line of the coins with a given velocity. Find the velocity with which the last coin will start to move.

31. A ball of mass m, which is at rest on a smooth horizontal plane, is tied by means of a string to a fixed point at the same height as the center of the ball. A second ball of equal radius but of mass m' is projected along the plane with a velocity $\mathbf{v}$ which makes an angle α with the string. The second ball collides with the first centrally and gives it a velocity u. Show that

$$u = \frac{m' \sin \alpha \,(1 + e)}{m + m' \sin^2 \alpha}\, v.$$

CHAPTER XIII.

ANGULAR IMPULSE AND ANGULAR MOMENTUM.

206. Angular Impulse. — The mechanical results produced by a torque may be measured in two ways. If the torque is considered to act through an angle the result measured is the work done by the torque; on the other hand if the torque is considered to act during an interval of time the result measured is called *angular impulse.*

The angular impulse which a constant torque imparts to a body in an interval of time equals the product of the torque by the interval of time. If $\mathbf{H}$ denotes the angular impulse, $\mathbf{G}$ the torque, and t the time of action, then

$$\mathbf{H} = \mathbf{G} \cdot t. \tag{I$'$}$$

When a vector is multiplied by a scalar the product is a vector which has the same direction as the original vector. Therefore $\mathbf{H}$ is a vector and has the same direction as $\mathbf{G}$.

When torque is not constant angular impulse equals the vector sum of infinitesimal impulses imparted during infinitesimal intervals of time. Therefore

$$\left.\begin{aligned} \mathbf{H} &= \int_0^t \mathbf{G}\, dt \\ &= \int_0^t I\, \dot{\omega}\, dt \\ &= \int_{\omega_0}^{\omega} I\, d\omega, \end{aligned}\right\} \tag{I}$$

where ω_0 and ω are the angular velocities at the beginning and at the end of the interval of time during which the torque acts.

When I, the moment of inertia, remains constant, as in the case of a rigid body rotating about a fixed axis, the last integration can be performed at once and the following result obtained:

$$\int_0^t \mathbf{G}\,dt = I\boldsymbol{\omega} - I\boldsymbol{\omega}_0. \tag{II}$$

207. Angular Momentum.—The magnitude $I\boldsymbol{\omega}$ is called *angular momentum* and is defined as the product of the moment of inertia by the angular velocity. Since I is a scalar $I\boldsymbol{\omega}$ is a vector which has the same direction as $\boldsymbol{\omega}$. Equation (II) states that *angular impulse equals the change in the angular momentum.*

208. Moment of Momentum.—Angular momentum is often called *moment of momentum*, because the former may be considered as the moment of the linear momenta of the particles of the system under consideration. Let dm be an element of mass, r its distance from the axis of rotation, and v its linear velocity. Then the moment of the momentum of dm about the axis is $r \cdot v\,dm$. Therefore the total moment of momentum is

$$\begin{aligned}\int_0^m r \cdot v\,dm &= \int_0^m r \cdot r\omega\,dm \\ &= \int_0^m r^2\,dm \cdot \omega \\ &= I\omega,\end{aligned}$$

which is the angular momentum.

209. Dimensions and Units.—Substituting the dimensions of $\mathbf{G}$, t, I, and $\boldsymbol{\omega}$ in equation (II) we find that both angular impulse and angular momentum have the dimensions $[ML^2T^{-1}]$. The units are also the same for both. The C.G.S. unit is $\frac{\text{gm. cm.}^2}{\text{sec.}}$ and the British unit is ft. lb. sec.

210. Torque and Angular Momentum.—When the moment of inertia of a body remains constant under the action of a torque we have

$$\mathbf{G} = I\frac{d\boldsymbol{\omega}}{dt}$$
$$= \frac{d}{dt}(I\boldsymbol{\omega}). \qquad \text{(III)}$$

Therefore *torque equals the time rate of change of momentum.*

The following analysis proves that the last statement is true when the moment of inertia varies with the time as well as when it remains constant.

Let A, Fig. 122, represent a body, or a system of bodies, which is acted upon by one or more external torques. For the sake of simplicity suppose the planes of the torques to be parallel to the plane of the paper, and the axis of rotation to pass through the point O and to be perpendicular to the plane of the paper. Let $d\mathbf{F}$ be the resultant force acting upon an element of mass dm. Then the moment of $d\mathbf{F}$ about the axis of rotation equals the product of r, the distance of dm from the axis, by $d\mathbf{F}_p$, the component of $d\mathbf{F}$ perpendicular to r. Therefore

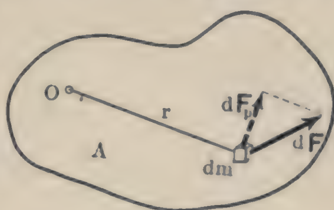


Fig. 122.

$$dG = r \cdot dF_p$$
$$= r \cdot dmf_p$$
$$= r \cdot dm \cdot \frac{1}{r}\frac{d}{dt}(r^2\omega) \qquad \text{[p. 97]}$$
$$= dm \cdot \frac{d}{dt}(r^2\omega)$$
$$= \frac{d}{dt}(r^2\,dm \cdot \omega).$$

Therefore the resultant external torque acting upon the body is

$$G = \frac{d}{dt}\left(\omega \int_0^m r^2\, dm\right)$$
$$= \frac{d}{dt}(I\omega),$$

or

$$\mathbf{G} = \frac{d}{dt}(I\boldsymbol{\omega}), \qquad \text{(III)}$$

where I is supposed to vary with the time. Equation (III) is the general form of torque equation, of which equation (V) of Chapter XI is a special case.

Introducing this expression of $\mathbf{G}$ in the definition for angular impulse we obtain

$$\mathbf{H} = \int_0^t \mathbf{G}\, dt$$
$$= \int_0^t \frac{d}{dt}(I\boldsymbol{\omega})\, dt$$
$$= \int_0^t d(I\omega)$$
$$= I\boldsymbol{\omega} - I_0\boldsymbol{\omega}_0, \qquad \text{(IV)}$$

where I_0 and $\boldsymbol{\omega}_0$ denote the moment of inertia and the angular acceleration at the instant $t = 0$, and I and $\boldsymbol{\omega}$ those at $t = t$. Equation (IV) is a generalization of equation (II). It states that *angular impulse equals the change in the angular momentum under all circumstances.*

211. The Principle of the Conservation of Angular Momentum.— When the resultant external torque acting upon a body or system of bodies vanishes, it follows from equation (III) that

$$\frac{d}{dt}(I\boldsymbol{\omega}) = 0,$$

and consequently

$$I\boldsymbol{\omega} = \text{const.} \qquad \text{(V)}$$

Therefore *if the resultant of all the external torques acting upon a system vanishes, the angular momentum of the system*

remains constant, in direction as well as in magnitude. This is the principle of the conservation of angular momentum.

ILLUSTRATIVE EXAMPLE.

Discuss the effect of a shrinkage in the radius of the earth upon the length of the day.

Let P and P' be the lengths of the day when the radius of the earth is a and a', respectively. Further, let ω and ω' be the corresponding values of the angular velocity of the earth about its axis. Then

$$\frac{P'}{P} = \frac{\omega}{\omega'}. \tag{1}$$

But since the earth is not supposed to be acted upon by any external torques its angular momentum remains constant. Therefore

$$I\omega = I'\omega'. \tag{2}$$

From equations (1) and (2) we obtain

$$\frac{P'}{P} = \frac{I'}{I} = \frac{a'^2}{a^2},$$

or

$$\frac{P - P'}{P} = \frac{a^2 - a'^2}{a^2}$$

and

$$\frac{\delta P}{P} = \frac{a + a'}{a} \cdot \frac{\delta a}{a}, \tag{3}$$

where δP and δa denote the diminutions in the length of the day and the radius, respectively. When δa is small a' is very nearly equal to a, therefore equation (3) may be written in the form

$$\frac{\delta P}{P} = 2\frac{\delta a}{a}. \tag{4}$$

Therefore the percentage diminution in the length of the day is twice as large as the percentage diminution in the radius. Hence when the radius is diminished by 1 mile the length of the day is diminished by about 43 seconds.

PROBLEMS.

1. How do the oceanic currents from the polar regions affect the length of the day?

2. A uniform rod of negligible diameter falls from a vertical position with its lower end on a perfectly smooth horizontal plane. What is the path of its middle point?

3. While passing through the tail of a comet an amount of dust of mass m settles uniformly upon the surface of the earth. Find the consequent change in the length of the day.

4. In the preceding problem find the torque due to the addition of mass. Suppose the passage to take n days and the rate at which mass is acquired to be constant.

5. A particle revolves, on a smooth horizontal plane, about a peg, to which it is attached by means of a string of negligible mass. The string winds around the peg as the particle rotates. Discuss the motion of the particle.

6. A mouse is made to run around the edge of a horizontal circular table which is free to rotate about a vertical axis through the center. Find the velocity of the mouse relative to the table which will give the latter 20 revolutions per minute? The table weighs 2 pounds and has a diameter of 18 inches; the mouse weighs 5 ounces.

7. In the preceding problem find the velocity of the mouse with respect to the ground.

8. A cylindrical vessel of radius a is filled with a liquid, closed tight, and made to rotate with a constant angular velocity ω_0 about its geometrical axis, which is vertical. Suppose the frictional forces between the inner surface of the vessel and the liquid and between the molecules of the liquid to be small, yet enough to transmit the motion to the liquid if the rotation is kept up for a long time. After each particle of water attains an angular velocity about the axis given by the relation $\omega = \omega_0 r$ the torque which kept the angular velocity constant is stopped and the liquid is suddenly frozen. What will be the angular velocity of the system if

(a) The mass of the vessel is negligible.

(b) The mass is not negligible but the thickness is. Take the ends into account.

(c) Neither the mass nor the thickness of the cylinder is negligible. Do not take the ends into account.

(d) In (c) take the ends into account.

9. In the preceding problem suppose the distribution of the angular velocity of the liquid about the axis just before it is frozen to be given by the relation $\omega = \omega_0 e^{\frac{a-r}{r}}$, where r is the distance from the axis.

APPLICATION TO SPECIAL PROBLEMS.

212. Ballistic Pendulum.—A ballistic pendulum is a heavy target which is used to determine the velocity of projectiles. The target, which is suspended from a horizontal axis, is given an angular displacement when it receives the projectile. Considering the target and the bullet which is projected into it as an isolated system we apply the principles of the conservation of energy and of the conservation of angular momentum. Just before the bullet hits the target the angular momentum of the system about the axis is that due to the velocity of the bullet and equals $I'\frac{v}{b}$, where I' is the moment of inertia of the bullet about the axis, v is its velocity, and b is its distance from the axis just before it hits the target, Fig. 123. The bullet is supposed to hit the target normally, when the latter is in the equilibrium position, and to be imbedded in it. The angular momentum just after the bullet hits the target is $(I + I')\,\omega$, where I is the moment of inertia of the target and ω its initial angular velocity. Then, by the conservation of the angular momentum, we have

FIG. 123.

$$I'\frac{v}{b} = (I + I')\,\omega. \quad \therefore v = b\,\frac{I + I'}{I'}\,\omega. \tag{1}$$

If we suppose the energy lost during the impact to be negligible the kinetic energy of rotation just after the bullet hits the target equals the potential energy of the system at its position of maximum angular displacement. Therefore

$$\tfrac{1}{2}\,(I + I')\,\omega^2 = (M + m)\,ga\,(1 - \cos\alpha), \tag{2}$$

where M and m are the masses of the target and of the bullet, respectively, a is the distance of the center of mass of the system from the axis, and α is the maximum angular dis-

placement. Eliminating ω between equations (1) and (2) we obtain

$$v = \frac{b}{I'}\sqrt{2\,ga\,(I + I')\,(M + m)\,(1 - \cos\alpha)}. \tag{3}$$

The moment of inertia of the target may be determined by observing the period of oscillation when it is used as a pendulum. It will be shown later* that if P denotes the period then

$$P = 2\,\pi\sqrt{\frac{I + I'}{(M + m)\,ga}}. \tag{4}$$

Eliminating $(I + I')$ between equations (3) and (4) we get

$$\left.\begin{aligned} v &= \frac{Pabg\,(M + m)}{\pi I'}\sqrt{\frac{1 - \cos\alpha}{2}} \\ &= \frac{Pabg\,(M + m)}{\pi I'}\sin^2\frac{\alpha}{2}. \end{aligned}\right\} \tag{5}$$

But in practice m is very small compared with M, the bullet is small enough to be considered as a small particle, and α is small; therefore we can neglect m in the numerator, substitute mb^2 for I', and replace $\sin\frac{\alpha}{2}$ by $\frac{\alpha}{2}$. When these simplifications are introduced into equation (5) we get

$$v = \frac{PagM}{4\,\pi mb}\,\alpha^2. \tag{6}$$

213. Motion Relative to the Center of Mass.—Suppose a rigid body to have a uniplanar motion. Let M be the mass of the body, I its moment of inertia with respect to an axis perpendicular to the plane of the motion, I_c its moment of inertia about a parallel axis through the center of mass, and a the distance between the two axes. Then the angular momentum about the first axis is

$$\left.\begin{aligned} I\omega &= (I_c + Ma^2)\,\omega \\ &= I_c\omega + a\cdot M\bar{v}, \end{aligned}\right\} \tag{VI}$$

where $\bar{v}$ is the velocity of the center of mass. In the right hand member of the last equation the first term represents

* Page 309.

the angular momentum of the body due to the motion of its particles relative to the center of mass, while the second term represents the angular momentum of the body due to the motion of its particles *with* the center of mass. The second term depends upon the position of the center of mass relative to the axis of rotation. The first term does not at all depend upon this position. It depends upon the distribution of the particles of the body about the center of mass. The two terms are, therefore, independent; that is, if the center of mass of a body is suddenly fixed the angular momentum of the body due to the motion of its particles about the center of mass is not at all affected. On the other hand if the motion about the center of mass is destroyed the angular momentum about a given axis due to the motion of the particles of the body *with* the center of mass is not changed. In other words *motion about the center of mass and motion with the center of mass are distinct and independent.**

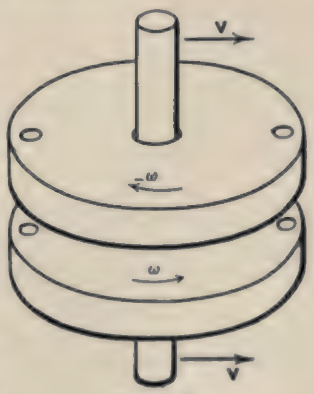


FIG. 124.

As an illustration of this important fact consider two disks, Fig. 124, of equal mass, radius, and thickness, which have equal and opposite angular velocities about a common axle, and which move with the axle in a direction perpendicular to it. Suppose each of the disks to have two similarly placed holes, as shown in the figure, so that they can be made one solid piece by dropping a pin in each pair of holes when they are in line. If the rotational motion is stopped by dropping the pins into the holes, the motion of the axle goes on as if nothing had happened. On the other hand if the motion with the axle is changed or even stopped, the rotations of the disks about the axle are not at all disturbed.

* This result holds true for all bodies and systems, whether rigid or not.

ILLUSTRATIVE EXAMPLE.

A uniform circular hoop rotates about a peg on a perfectly smooth horizontal plane; find the angular velocity of the hoop if the peg is suddenly removed and simultaneously another peg is introduced, about which it begins to rotate.

Let O and O', Fig. 125, be the positions of the first and second peg, respectively. The circle in continuous line may be considered to represent the position of the hoop just before it stops rotating about O and just after it begins to rotate about O'.

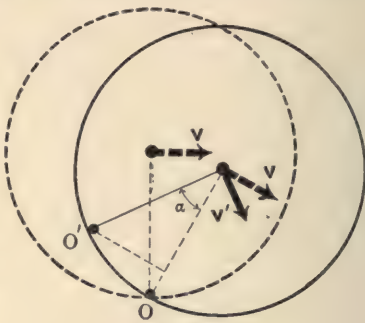


FIG. 125.

The only force which comes into play when the hoop strikes the peg O' passes through O', hence it produces no effect upon the angular momentum about O'. Therefore the angular momentum about O' just after the hoop strikes the peg equals the angular momentum just before. The angular momentum after the hoop begins to rotate about O' is

$$H'_{o'} = I\omega' = 2\,ma^2\omega',$$

where $H'_{o'}$ is the angular momentum and ω' the angular velocity about the O', m the mass, and a the radius of the hoop.

The angular momentum about O' just before the hoop begins to rotate about O' equals the angular momentum of the hoop due to the motion of the hoop about its geometrical axis plus its angular momentum due to its motion *with* its center of mass. Therefore

$$\begin{aligned} H_{o'} &= I_c\,\omega + mv \cdot a\cos\alpha \\ &= ma^2\omega + ma^2\omega\cos\alpha \\ &= ma^2\omega\,(1 + \cos\alpha), \end{aligned}$$

where ω is the angular velocity about the peg O, and α the angle which the arc OO' subtends at the center of the hoop. But since

$$H'_{o'} = H_{o'},$$

$$2\,ma^2\omega' = ma^2\omega\,(1 + \cos\alpha)$$

and

$$\omega' = \frac{1 + \cos\alpha}{2}\,\omega,$$

or

$$v' = \frac{1 + \cos\alpha}{2}\,v,$$

where v is the linear velocity of the center of the hoop while the latter rotates about O, and v' the velocity afterwards.

Discussion. — When $\alpha = 0$, that is, when the two pegs coincide, $\omega' = \omega$ and $v' = v$, as they should. When $\alpha = \frac{\pi}{2}$, $\omega' = \frac{\omega}{2}$, $v' = \frac{v}{2}$. When $\alpha = \pi$, $\omega' = 0$ and $v' = 0$, that is, the hoop comes to rest.

PROBLEMS.

1. A rod of negligible transverse dimensions and length l is moving on a smooth horizontal plane in a direction perpendicular to its length. Show that if it strikes an obstacle at a distance a from its center it will have an angular velocity equal to $\frac{3\,av}{l^2}$, where v is its linear velocity before meeting the obstacle.

2. A uniform circular plate is turning about its geometrical axis on a smooth horizontal plane. Suddenly one of the elements of its lateral surface is fixed. Show that the angular velocity after fixing the element equals $\frac{\omega}{3}$, where ω is the angular velocity before fixing it.

3. A circular plate which is rotating about an element of its lateral surface is made to rotate about another element by suddenly fixing the second and freeing the first. Show that $\omega' = \frac{1 + 2\cos\alpha}{3}\omega$, where ω and ω' are the values of the angular velocity of the plate before and after fixing the second element, and α is the angular separation of the two elements when measured at the center of the disk.

4. Three particles of equal mass are attached to the vertices of an equilateral triangular frame of negligible mass. Show that if one of the vertices is fixed while the frame is rotating about an axis through the center of the triangle perpendicular to its plane the angular velocity is not changed.

5. A square plate is moving on a smooth horizontal plane with a velocity $\mathbf{v}$ at right angles to two of its sides. Find the velocity with which it will rotate if

(a) one of its corners is suddenly fixed;

(b) the middle point of one of its sides is fixed.

6. A uniform rod of negligible transverse dimensions is rotating about a transverse axis through one end. Find the angular uelocity with which it will rotate if the axis is suddenly removed and simultaneously a parallel axis is introduced through the center of mass of the rod.

7. An equilateral triangular plate is rotating about an axis through

one of the vertices perpendicular to the plane of the plate. Find the resulting angular velocity due to a sudden removal of the axis and a simultaneous introduction of a parallel axis through the center of mass.

8. In the preceding problem, suppose the new axis to pass through one of the other two vertices.

214. Reaction of the Axis of Rotation.—Suppose B, Fig. 126, to be a rigid body free to rotate about a fixed axis through the point O, perpendicular to the plane of the figure. If an external force $\mathbf{F}$ is applied to the body a part of its action is, in general, transmitted to the axis of rotation. This results in the reaction, $\mathbf{R}$, of the axis, which we will investigate. For the sake of simplicity suppose $\mathbf{F}$ to lie in the plane which passes through the center of mass, c, perpendicular to the axis.

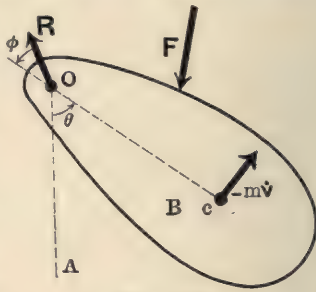


Fig. 126.

Since $\mathbf{F}$ and $\mathbf{R}$ are supposed to be the only external forces acting upon the body, then by equation (VIII) of p. 242

$$m\dot{\bar{\mathbf{v}}} = \mathbf{F} + \mathbf{R}, \tag{1}$$

where $\dot{\bar{\mathbf{v}}}$ is the acceleration of the center of mass. If $\mathbf{F}_n$ and $\mathbf{F}_\tau$ denote the components of $\mathbf{F}$ along and at right angles to the line Oc, respectively, and $\mathbf{P}$ and $\mathbf{Q}$ the components of $\mathbf{R}$ along the same directions, equation (1) may be resolved into the following component-equations:

$$m\bar{f}_n = F_n + P, \tag{2}$$

$$m\bar{f}_\tau = F_\tau + Q, \tag{3}$$

where $\bar{\mathbf{f}}_n$ and $\bar{\mathbf{f}}_\tau$ are the components of $\dot{\bar{\mathbf{v}}}$. But since the path of the center of mass is a circle

$$\bar{f}_n = \frac{\bar{v}^2}{a} = a\omega^2$$

and

$$\bar{f}_\tau = \dot{\bar{v}} = a\dot{\omega},$$

where a is the distance of the center of mass from the axis and ω the angular velocity of the body. Making these substitutions in equations (2) and (3) and solving for P and Q we obtain

$$P = -F_n + ma\omega^2, \tag{VII}$$

$$Q = -F_t + ma\dot{\omega}. \tag{VIII}$$

The magnitude and the direction of **R** are given by the relations

$$R = \sqrt{P^2 + Q^2}$$

and

$$\tan\phi = \frac{Q}{P},$$

where ϕ is the angle **R** makes with the line Oc.

ILLUSTRATIVE EXAMPLE.

A uniform rod, which is free to rotate about a horizontal axis through one end, falls from a horizontal position. Find the reaction of the axis at any instant of its fall.

Evidently

$$F_n = -mg\cos\theta.$$

$$F_\tau = -mg\sin\theta.$$

The negative sign in the first equation is due to the fact that in equation (VII) F_n is supposed to be directed towards the axis, while $m\mathbf{g}\cos\theta$ is directed away from the axis. The negative sign in the second equation is due to the fact that θ is measured in the counter-clockwise direction, while $m\mathbf{g}\sin\theta$ points in the opposite direction.

Substituting these values of F_n and F_τ in equations (VII) and (VIII), we obtain

$$P = mg\cos\theta + ma\omega^2,$$

$$Q = mg\sin\theta + ma\dot{\omega}.$$

But by the conservation of energy

$$\tfrac{1}{2}I\omega^2 = mga\cos\theta,$$

where a is one-half the length of the rod. Therefore

$$\omega^2 = \frac{2\,mga}{I}\cos\theta = \frac{3\,g}{2\,a}\cos\theta$$

and

$$\dot{\omega} = -\frac{3\,g}{4\,a}\sin\theta.$$

Making these substitutions

$$P = \tfrac{5}{2}\, mg \cos\theta.$$

$$Q = \tfrac{1}{4}\, mg \sin\theta.$$

$$R = \frac{mg}{4}\sqrt{1 + 99\cos^2\theta}.$$

$$\tan\phi = \tfrac{1}{10}\tan\theta.$$

Discussion. — The reaction and its direction are independent of the length of the rod. When $\theta = 0$, $Q = 0$ and $R = P = \frac{5}{2}\, mg$. In other words at the instant when the rod passes the lowest point the force on the axis is $\frac{5}{2}$ times as large as the force when rod hangs at rest. When $\theta = \frac{\pi}{2}$, $P = 0$ and $R = Q = \frac{1}{4}\, mg$. If the rod is held in a horizontal position by supporting the free end the reaction of the axis is $\frac{1}{2}\, mg$. But as soon as the support is removed from the free end the reaction on the axis is changed from $\frac{1}{2}\, mg$ to $\frac{1}{4}\, mg$.

PROBLEMS.

1. A uniform rod which is free to rotate about a horizontal axis falls from the position of unstable equilibrium. Find the reaction of the axis.

2. In the preceding problem find the position where the horizontal component of the reaction is a maximum.

3. A uniform rod which is free to rotate about a horizontal axis falls from a horizontal position. Show that the horizontal component of the reaction is greatest when the rod makes 45° with the vertical.

4. A cube rotates about a horizontal axis which coincides with one of its edges. If at the highest position it barely completes the revolution, show that $P = \dfrac{3 - 5\cos\theta}{2} W$ and $Q = \dfrac{\sin\theta}{4} W$, where W is the weight of the cube.

5. A cube which is free to rotate about a horizontal axis through one of its edges starts to fall when its center is at the same level as the axis of rotation. Find the reaction of the axis.

6. Show that if the body of § 214 is a particle connected to the axis with a massless rod the reaction perpendicular to the rod vanishes.

7. Consider the reactions of the axis when the latter passes through the center of mass of the rigid body.

8. A circular plate is free to rotate about a horizontal axis which forms one of the elements of its cylindrical surface. The plate is let fall from the position when its center of mass is vertically above the axis. Determine the reaction of the axis at $\theta = \frac{\pi}{2}$ and at $\theta = 0$.

9. A hoop barely completes rotations about a horizontal axis which passes through its rim and is perpendicular to its plane. Determine the reaction of the axis at the lowest and the highest positions.

10. A uniform rod which rotates about a horizontal axis through one end has four times as much kinetic energy as it has potential energy at the instant it passes the highest point. Find the reaction of the axis when the rod is

(a) at the highest position;
(b) horizontal;
(c) at the lowest position.

215. Impulsive Reaction of an Axis. Center of Percussion.— If a rigid body which is free to rotate about a fixed axis is so struck that no impulse is imparted to the axis during the blow, any point of the line of action of the blow is called a *center of percussion* for that axis. It is evident that if the axis be removed and the blow applied at a center of percussion which corresponds to the removed axis, the body will rotate as if the axis were not removed. The axis about which a free rigid body rotates when it is given a blow is called the *axis of spontaneous rotation*.

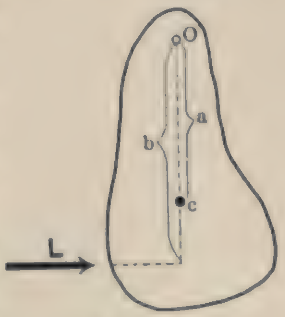


FIG. 127.

Suppose the rigid body of Fig. 127 to be free to rotate about an axis through O perpendicular to the plane of the figure. For the sake of simplicity suppose the blow to be applied in such a direction that it tends to produce rotation only about the given axis. Let **L** denote the linear impulse of the blow and **L**′ the impulse given to the body by the reaction of the axis of rotation. Then by the conservation of linear momentum the linear momentum of the body must be equal to the impulse given to it by the blow and by the reaction of the axis. Therefore

$$mv = L + L', \tag{1}$$

where m is the mass of the body and v the velocity of its center of mass. But by the conservation of angular momentum the angular momentum of the body about the axis after the blow must equal that of the blow itself. Therefore

$$I\omega = Lb, \tag{2}$$

where I is the moment of inertia of the body, ω its angular velocity and b the distance of the line of action of the blow from the axis.

Eliminating L between equations (1) and (2) and solving for L' we obtain

$$L' = mv - \frac{I\omega}{b}, \tag{3}$$

$$= \left(ma - \frac{I}{b}\right)\omega, \tag{IX}$$

where a is the distance of the center of mass from the axis of rotation. Equation (IX) gives the impulse produced by the reaction of the axis.

If the blow is applied at a center of percussion $L' = 0$. Therefore

$$ma - \frac{I}{b} = 0$$

and

$$b = \frac{I}{ma}. \tag{X}$$

PROBLEMS.

1. A square plate is moving on a smooth horizontal plane with two of its sides parallel to the direction of motion. Find the angular velocity with which it will rotate, also the impulsive reaction of the axis,

(a) if one of the corners is fixed;

(b) if the middle point of one of the sides is fixed.

2. An equilateral triangular plate is moving on a smooth horizontal plane in a direction perpendicular to one of its sides. Find the resulting angular velocity, also the impulse given by the axis,

(a) if one of its corners is fixed;

(b) if the middle point of one of its sides is fixed.

3. A hoop is moving on a smooth horizontal plane with its axis perpen-

dicular to the plane. Suppose a point on it to be fixed and find expressions for the resulting angular velocity and impulse imparted. Discuss the expressions for special positions of the fixed point.

4. While a circular plate is moving on a smooth horizontal plane one of the elements of its lateral surface is fixed. Find expressions for the resulting angular velocity and the impulse given by the axis. Discuss the results for special positions of the axis of rotation.

5. A uniform rod lies on a smooth horizontal plane. Where must a blow be struck so that it rotates about one end?

6. In the preceding problem can the rod be made to rotate about its middle point by a single blow?

7. A circular plate which lies on a smooth horizontal plane is struck so that it rotates about one of the elements of its lateral surface as an axis. Find the position where the blow is applied.

8. Find the center of percussion of a hoop which is free to rotate about an axis perpendicular to its plane.

9. How must a triangular plate, placed on a smooth horizontal plane, be struck so that it may rotate about one of its vertices?

GENERAL PROBLEMS.

1. Two particles of equal mass are connected by a string of length l and of negligible mass and placed on a smooth horizontal table so that one of the particles is near an edge of the table and the string is stretched at right angles to the edge. The particle near the edge is given a small displacement so that it begins to fall. Show that the interval of time between the instant at which the second particle leaves the table and the instant at which the string occupies a horizontal position is given by

$$t = \frac{\pi}{2}\sqrt{\frac{l}{g}}.$$

2. A uniform bar of negligible cross-section, which is rotating on a smooth horizontal plane about a vertical axis, strikes an obstacle and begins to rotate in the opposite direction. If L and L' denote the impulses given by the collision to the axis and the obstacle, respectively, ω and ω' the angular velocities of the bar before and after the collision, l the length and m the mass of the bar, and a the distance of the obstacle from the axis, show that

(a) $\omega' = e\omega$;

(b) $L = m(1+e)\dfrac{l(7l-6a)}{12a}\omega$;

(c) $L' = m(1+e)\dfrac{7l^2}{12a}\omega$.

3. A circular table is perfectly free to rotate about a vertical axis through its center. Show that if a man walks completely around the edge of the table the latter turns through an angle of $\frac{m}{M + 2m} \cdot 2\pi$, where m and M are the masses of the table and of the man, respectively.

4. A circular plate is rotating about its axis, which is vertical, with an angular velocity ω and is moving on a smooth horizontal plane with a linear velocity v. Find the angular velocity it will have if one of the elements of its lateral surface is suddenly fixed, and determine the impulse given to the axis of rotation. Discuss the results for special positions of the fixed axis.

5. A uniform rod strikes at one end against an obstacle while falling transversely. Show that the impulse which the obstacle receives will be one-half that which it would have received if the other end of the rod had struck an obstacle simultaneously with the first.

6. A particle is projected into a tube which is bent to form a circle and is lying on a smooth horizontal table. If the inner surface of the tube is perfectly smooth, show that the center of mass of the two moves in the direction of projection of the particle with a velocity of $\frac{m}{m + M} v$, while the particle and the center of the tube describe circles about it with an angular velocity $\frac{v}{a}$, where M is the mass and a the radius of the tube, while m is the mass and v the velocity of projection of the particle.

7. Find the direction and point of application which an impulse must have in order to make a sphere rotate about a tangent.

8. A uniform rod which is rotating on a smooth horizontal plane about a pivot through its middle point breaks into two equal parts. Determine the subsequent motion.

9. A uniform rod rotates on a smooth horizontal plane about a pivot. What will be the motion when the pivot breaks?

10. A uniform rod falls from a position where its lower end is in contact with a rough horizontal plane with which it makes an angle α. Show that when it becomes horizontal its angular velocity is $\sqrt{\frac{3 g \sin \alpha}{l}}$, where l is the length of the rod.

11. Show that in problem (10) the angular velocity will be the same when the horizontal plane is smooth.

12. A uniform rod which lies on a smooth horizontal plane is struck at one end, transversely. Show that the energy imparted equals $\frac{4}{3}$ of the energy which would have been given to the bar by the same blow if the other end of the bar were fixed.

CHAPTER XIV.

MOTION OF A PARTICLE IN A CENTRAL FIELD OF FORCE.

216. Central Field of Force. — A region is called a central field of force when the intensity of the field at every point of the region is directed toward a fixed point. The fixed point is called the *center of the field.* The force which a particle experiences when placed in a central field of force is called a *central force.*

217. Equations of Motions. — Consider the motion of a particle which is projected into a central field of force. It is evident from symmetry that the path will lie in the plane determined by the center of the field and the direction of projection. The expressions for the radial and transverse components of the acceleration are, according to the results of § 90,

$$f_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2,$$

$$f_p = \frac{1}{r}\frac{d}{dt}(r^2\omega).$$

When the center of the field is chosen as the origin the force acts along the radius vector. Therefore the transverse acceleration vanishes. Suppose the force and the acceleration to be functions of the distance of the particle from the center, then the last two equations become

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -f(r), \qquad \text{(I)}$$

$$\frac{d}{dt}(r^2\omega) = 0. \qquad \text{(II)}$$

where $-f(r)$ is the total acceleration. The negative sign in the right-hand member of equation (I) indicates the fact that the acceleration is directed toward the center, while the radius vector is measured in the opposite direction. Equations (I) and (II) are the differential equations of the motion of a particle in a central field of force.

218. General Properties of Motion in a Central Field.—Integrating equation (II) we get

$$r^2\omega = h, \tag{III}$$

where h is a constant. The following properties, which are direct consequences of equation (III), are common to all motions in central fields of force.

(1) The radius vector sweeps over equal areas in equal intervals of time.

When the radius vector turns through an angle $d\theta$ it sweeps over an area equal to $\frac{1}{2}r \cdot r\,d\theta$; therefore the rate at which the area is described equals

$$\frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\omega = \frac{1}{2}h = \text{constant.}$$

(2) The angular velocity of the particle varies inversely as the square of the distance of the particle from the center of force. This is evident from equation (III).

(3) The linear velocity of the particle varies inversely as the length of the perpendicular which is dropped upon the direction of the velocity from the center of force.

It was shown on page 87 that

$$\omega = \frac{v\cos\phi}{r},$$

where v is the linear velocity and ϕ the angle which the velocity makes with a line perpendicular to the radius vector. Let p denote the length of the perpendicular dropped from

the center of force upon the direction of the velocity; then it is evident from Fig. 128 that

$$\cos \phi = \frac{p}{r}.$$

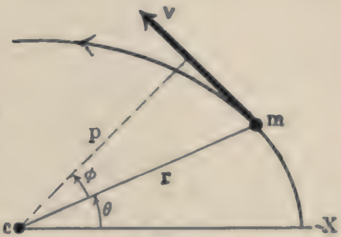


FIG. 128.

Substituting this value of $\cos \phi$ in the preceding equation we obtain

$$\omega = \frac{pv}{r^2},$$

or

$$v = \frac{r^2\omega}{p} = \frac{h}{p}. \tag{IV}$$

(4) The angular momentum of the particle with respect to the center remains constant.

This result is obtained at once by multiplying both sides of equation (III) by m, the mass of the particle. Thus

$$mr^2\omega = mh,$$

but

$$mr^2\omega = I\omega.$$

Therefore

$$I\omega = mh = \text{constant}.$$

219. Equation of the Orbit. — The general equation of the orbit is found by eliminating t between equations (I) and (III). The analytical reasoning which follows does not need further explanation:

$$\frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} = \omega \frac{dr}{d\theta}$$

$$= \frac{h}{r^2} \frac{dr}{d\theta} \qquad \text{[by (III)]}$$

$$= -h \frac{d\left(\frac{1}{r}\right)}{d\theta}$$

$$= -h \frac{du}{d\theta},$$

where $u = \frac{1}{r}$. Therefore

$$\frac{d^2r}{dt^2} = -h\frac{d^2u}{d\theta^2}\cdot\frac{d\theta}{dt}$$

$$= -h^2u^2\frac{d^2u}{d\theta^2}.$$

Substituting this value of $\frac{d^2r}{dt^2}$ and the value of $\frac{d\theta}{dt}$, which may be obtained from equation (III), in equation (I), we have

$$\frac{d^2u}{d\theta^2} + u = \frac{f(r)}{h^2u^2} \tag{V}$$

for the equation of the orbit. When the law of force is given $f(r)$ is known and the orbit is determined by equation (V). On the other hand if the orbit is given equation (V) determines the law of force. Thus, if F denotes the force,

$$F = -mf(r)$$

$$= -mh^2u^2\left(u + \frac{d^2u}{d\theta^2}\right). \tag{VI}$$

ILLUSTRATIVE EXAMPLE.

A particle describes a circle in a central field of force. Determine the law of force if the center of the field lies on the path.

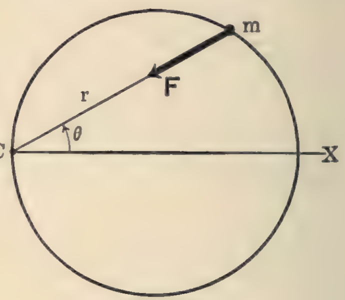


Fig. 129

Taking the center as the origin, Fig. 129, and the diameter through the origin as the axis and referring the circle to polar coördinates we obtain

$$r = 2a\cos\theta, \tag{1'}$$

or

$$u = \frac{1}{2a\cos\theta} \tag{1}$$

for the equation of the orbit. Differentiating the last equation

$$\frac{d^2u}{d\theta^2} = \left(\frac{1}{a\cos^3\theta} - \frac{1}{2a\cos\theta}\right)$$

$$= 8a^2u^3 - u. \tag{2}$$

Substituting in equation (VI) from equations (1) and (2) we get

$$F = -\frac{8a^2h^2m}{r^5}. \tag{3}$$

Therefore the force varies inversely as the fifth power of the distance from the center of force. The negative sign in the second member of equation (3) shows that the force is directed towards the origin; in other words, it is an attractive force.

PROBLEMS.

1. Show that if a particle describes the reciprocal spiral $r\theta = a$ in a central field of force, the force is attractive and varies inversely as the cube of the distance from the origin, which is the center of attraction.

2. Show that if a particle describes the logarithmic spiral $r = e^{a\theta}$ in a central field of force, the expression for the force is $F = -\frac{mh^2(a+1)}{r^3}$.

3. A particle moves in a central field of force where the force is away from the center and is proportional to the distance. Show that the orbit is a hyperbola.

4. Show that in the preceding problem the radius vector sweeps over equal areas in equal intervals of time.

5. A particle describes an ellipse in a field of force the center of which is at the center of the ellipse. Show that the force varies directly as the distance and is directed towards the center.

6. In the preceding problem show that the radius vector sweeps over equal areas in equal intervals of time.

7. A particle describes an ellipse in a field of force, the center of which is at one focus. Show that the force is towards the center of force, and is inversely proportional to the square of the distance.

220. Motion of Two Gravitating Particles.—Suppose two particles of masses m and M to move under the action of their mutual gravitational attraction, as in the case of the sun and the earth or the earth and the moon. Then if r is the distance between the centers and γ the gravitational constant the mutual force of attraction is

$$F = -\gamma\frac{mM}{r^2}.$$

In order to fix our ideas let M be the mass of the sun and m the mass of the earth. Then the sun gives the earth

an acceleration $-\gamma \frac{M}{r^2}$, while the earth imparts to the sun an acceleration equal to $\gamma \frac{m}{r^2}$.

Suppose at any instant we impart to both the sun and the earth a velocity equal and opposite to that of the sun and apply an acceleration $-\frac{m}{r^2}$. This will bring the sun to rest and keep it at rest, without altering the motion of the earth relative to the sun.* This reduces the problem of the motion of the earth to that of a particle moving in a central field of force where the acceleration is

$$-f(r) = -\gamma \frac{M}{r^2} - \gamma \frac{m}{r^2}$$

$$= -\gamma \frac{M+m}{r^2}.$$

or
$$f(r) = \frac{\mu}{r^2} \tag{VII}$$

and
$$F = -\mu \frac{m}{r^2}, \tag{VIII}$$

where
$$\mu = \gamma (M + m).$$

Substituting from equation (VIII) in equation (V) we obtain

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} \tag{1}$$

for the equation of the orbit. Let $u' = u - \frac{\mu}{h^2}$, then the equation of the orbit takes the form

$$\frac{d^2u'}{d\theta^2} + u' = 0. \tag{2}$$

In order to integrate equation (2) let $v = \frac{du'}{d\theta}$,

* The acceleration of a particle relative to another moving particle is found by adding the negative of the acceleration of the second particle to that of the first.

then
$$\frac{d^2u'}{d\theta^2} = \frac{dv}{d\theta} = \frac{dv}{du'}v,$$

and
$$v\frac{dv}{du'} + u' = 0.$$

Separating the variable and integrating

$$v^2 = A^2 - u'^2.$$

$$\therefore \frac{du'}{d\theta} = -\sqrt{A^2 - u'^2}.$$

Integrating again

$$\cos^{-1}\frac{u}{A} = \theta + \delta$$

or $\quad u' = A \cos(\theta + \delta).$

Let $\quad u' = A$ when $\theta = 0$, then $\delta = 0$. Therefore

$$u' = A \cos\theta \tag{3}$$

is a solution of equation (2). Substituting the value of u' in equation (3),

$$u = \frac{\mu}{h^2} + A\cos\theta \tag{4}$$

and replacing u by its value

$$r = \frac{ep}{1 - e\cos\theta} \tag{5}$$

where
$$e = -\frac{h^2A}{\mu}, p = -\frac{1}{A}. \tag{6}$$

Equation (5) is the well-known equation of a conic section. Therefore the orbit is a conic section with an eccentricity equal to $-\frac{h^2A}{\mu}$.

The expression for the velocity at any point of the orbit may be obtained by substituting the values of $\frac{dr}{dt}$ and $\frac{d\theta}{dt}$, which

may be obtained from equations (5) and (III), respectively, in equation (3) of page 82. Thus

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2$$

$$= \frac{h^2}{p^2}\sin^2\theta + \frac{h^2}{r^2}. \tag{7}$$

Eliminating $\sin^2\theta$ between equations (5) and (7) we get

$$v^2 = \frac{h^2}{e^2p^2}(e^2 - 1) + \frac{2\,h^2}{epr}$$

$$= \frac{h^2}{e^2p^2}(e^2 - 1) + \frac{2\,k}{r}, \tag{8}$$

where $$k = \frac{h^2}{ep} = \text{const.}$$

Therefore $$v^2 - \frac{2\,k}{r} = \frac{h^2}{e^2p^2}(e^2 - 1) = \text{const.} \tag{9}$$

221. Conditions which Determine the Type of the Orbit. — Suppose a gravitating body to be projected into the field of another gravitating body, which acts as the center of force; then the type of the orbit is determined by the initial conditions, that is, the magnitude and the direction of the velocity of projection and the distance of the particle from the center of force at the instant of projection. Substituting the initial values of v and r in equation (9) and rearranging we obtain the following expression for the eccentricity:

$$e^2 = 1 + \frac{e^2p^2}{h^2}\left(v_0{}^2 - \frac{2\,k}{r_0}\right). \tag{10}$$

The character of the orbit is determined by the value of the factor in the parentheses of equation (10). When it vanishes e is one, therefore the orbit is a parabola; when it is negative e is less than one, therefore the orbit is an ellipse; and when it is positive e is greater than one, therefore the orbit is a hyperbola. We have, therefore, the following criteria:

Case I. The orbit is a parabola, when $v_0^2 = \dfrac{2k}{r_0}$.

Case II. The orbit is an ellipse, when $v_0^2 < \dfrac{2k}{r_0}$.

Case III. The orbit is a hyperbola, when $v_0^2 > \dfrac{2k}{r_0}$.

The general expression for the velocity, which is given by equation (9), may be put in the following special forms:

I. $v^2 = \dfrac{2k}{r}$, when the orbit is a parabola.

II. $v^2 = k\left(\dfrac{2}{r} - \dfrac{1}{a}\right)$, when the orbit is an ellipse.

III. $v^2 = k\left(\dfrac{2}{r} + \dfrac{1}{a}\right)$, when the orbit is a hyperbola.

The quantity a is the length of the semi-transverse axis.

222. Velocity from Infinity. — The velocity which the particle acquires in falling towards the center from a point infinitely distant from the center is called the *velocity from infinity*. This velocity may be computed from the energy equation. Thus

$$\begin{aligned}\tfrac{1}{2} mv^2 &= \int_\infty^r F\,dr \\ &= \int_\infty^r \mu \frac{m}{r^2}\,dr \\ &= \frac{\mu m}{r}.\end{aligned}$$

Therefore
$$v^2 = \frac{2\mu}{r}.$$

But the last equation is identical with the relation which gives the velocity of a particle moving in a parabolic path, therefore if a particle describes a parabolic orbit its velocity at any point of its orbit is equal to the velocity it would have acquired if it had started from infinity and arrived at that

point of the field of force. This fact enables us to state the conditions which determine the type of the orbit in the following forms:

I. When the velocity of projection equals the velocity from infinity the orbit is a parabola.

II. When the velocity of projection is less than the velocity from infinity the orbit is an ellipse.

III. When the velocity of projection is greater than the velocity from infinity the orbit is a hyperbola.

Thus if a comet starts from rest at an infinite distance from the sun and falls towards the sun its orbit will be a parabola. If it is projected towards the sun from an infinite distance its orbit will be a hyperbola. If it falls from rest, starting from a finite distance, its orbit will be an ellipse.

223. Period of Revolution. — From equation (III) we have

$$h = r^2\omega = r^2\frac{d\theta}{dt}.$$

$$\therefore \quad h\,dt = r \cdot r\,d\theta = 2\,dA,$$

where dA is the area swept over by the radius vector in the time dt. Therefore when the orbit is an ellipse the period of revolution is

$$\begin{aligned} P &= \int_0^P dt \\ &= \frac{2}{h}\int_0^{\pi ab} dA \qquad (\pi ab = \text{area of ellipse}) \\ &= \frac{2\,\pi ab}{h}, \end{aligned}$$

where a and b are the semi-major axis and semi-minor axis of the ellipse, respectively. But by equations (6)

$$h = \sqrt{ep \cdot \mu}$$

and by the properties of the ellipse $ep = \dfrac{b^2}{a}$, therefore $h = \sqrt{\dfrac{b^2}{a}\mu}$.

Substituting the last expression for h in that for P we obtain

$$P = \frac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}}$$

$$= \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\gamma(M+m)}}.$$

It will be noted that the period of revolution depends upon the major axis but not upon the minor axis of the orbit.

The results obtained in discussing the motion of two gravitating particles are as they appear to an observer who is located on one of the bodies. The form and size of the orbit, the period of revolution, etc., will be the same whether the observer is located on one or on the other of the two bodies. For instance, to an observer on the moon the earth describes an orbit which is exactly similar to the orbit which the moon appears to describe to an observer on the earth.

224. Mass of a Planet which has a Satellite. — In order to fix our ideas let the earth be the planet. Then, since the acceleration due to the sun is practically the same on the moon as it is on the earth, the period of revolution of the moon around the earth is the same as if they were not in the gravitational field of the sun. Therefore the period of the moon around the earth is

$$P' = \frac{2\pi a'^{\frac{3}{2}}}{\sqrt{\gamma(m+m')}},$$

while that of the earth around the sun is

$$P = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\gamma(M+m)}},$$

where M, m, and m' are the masses of the sun, of the earth, and of the moon, respectively, a is the semi-major axis of the earth's orbit, and a' that of the moon's orbit.

Squaring these equations and dividing one by the other

$$\frac{m+m'}{M+m} = \left(\frac{P}{P'}\right)^2 \cdot \left(\frac{a'}{a}\right)^3.$$

Since m' is negligible compared with m, and m compared with M, the last equation may be written in the form

$$\frac{m}{M} = \left(\frac{P}{P'}\right)^2 \cdot \left(\frac{a'}{a}\right)^3,$$

which gives the ratio of the mass of the planet to that of the sun.

225. Kepler's Laws. — In establishing the truth of the law of gravitation Newton showed that the same law which makes the apple fall to the ground keeps the moon in its orbit. Then he extended the application of the law to the other members of the solar system by accounting for the empirical laws which Kepler (1571–1630) had formulated from the observations of Tycho Brahe (1546–1601). The following are the usual forms in which Kepler's laws are stated.

1. Each planet describes an ellipse in which the sun occupies one focus.
2. The radius vector describes equal areas in equal intervals of time.
3. The square of the period of any planet is proportional to the cube of the major axis of its orbit.

The first law is, as we have seen, a direct consequence of the inverse square law.

The second law follows from equation (III), which holds good for all bodies moving in central fields of force.

The third law amounts to stating that the masses of the planets are negligible compared with the mass of the sun. For if m, a, and P refer to one planet and m', a', and P' to another planet, then

$$P = \frac{2\,\pi a^{\frac{3}{2}}}{\sqrt{\gamma\,(M+m)}} \text{ and } P' = \frac{2\,\pi a'^{\frac{3}{2}}}{\sqrt{\gamma\,(M+m')}}.$$

Therefore

$$\left(\frac{P}{P'}\right)^2 = \left(\frac{a}{a'}\right)^3 \cdot \frac{M+m'}{M+m}.$$

Evidently when m and m' are negligible compared with M

$$\left(\frac{P}{P'}\right)^2 = \left(\frac{a}{a'}\right)^3,$$

which is of Kepler's third law.

PROBLEMS.

1. The gravitational acceleration at the surface of the earth is about 980 cm./sec.^{2} Calculate the mass and the average density of the earth, taking 6.4×10^8 cm. for the mean radius, and supposing it to attract as if all its mass were concentrated at its center.

2. The periods of revolution of the earth and of the moon are, roughly, $365\frac{1}{4}$ and $27\frac{1}{3}$ days. Find the mass of the moon in tons. Take 6.0×10^{27} gm. for the mass of the earth.

3. The periods of revolution of the earth and of the moon are $365\frac{1}{4}$ and $27\frac{1}{3}$ days, respectively, and the semi-major axes of their orbits are, approximately, 9.5×10^7 and 2.4×10^5 miles. Find the ratio of the mass of the sun to that of the earth.

4. Taking the period of the moon to be $27\frac{1}{3}$ days, and the radius of its orbit to be 3.85×10^{10} cm., show that the acceleration of the moon, due to the attraction of the earth, is equal to what would be expected from the gravitational law. Assume the gravitational acceleration at the surface of the earth, that is, at a point 6.4×10^8 cm. away from the center, to be $980 \dfrac{\text{cm.}}{\text{sec.}^2}$.

5. Show that if the earth were suddenly stopped in its orbit it would fall into the sun in about 62.5 days.

6. Show that if a body is projected from the earth with a velocity of 7 miles per second it may leave the solar system.

GENERAL PROBLEMS.

1. Find the expression for the central force under which a particle describes the orbit $r^n = a^n \cos n\theta$ and consider the special cases when

(a) $n = \frac{1}{2}$,	(c) $n = 1$,	(e) $n = 2$.
(b) $n = -\frac{1}{2}$,	(d) $n = 2$,	

2. A particle moves in a central field of force with a velocity which is inversely proportional to the distance from the center of the field. Show that the orbit is a logarithmic spiral.

3. A gun can project a shot to a height of $\frac{R}{n}$, where R is the radius of the earth. Taking the variation of the gravitational force with altitude, show that the gun can command $\frac{1}{n^2}$ of the earth's surface.

4. A particle is projected into a smooth horizontal circular groove. The particle is attracted towards a point in the radius which joins the position of projection with the center, with a force equal to $\frac{\mu}{r^2}$. Show that in order that the particle may be able to make complete revolutions the initial velocity must not be less than $\frac{4\,\mu b}{a^2 - b^2}$, where a is the radius of the groove and b the distance of the center of force from the center of the groove.

5. A comet describing a parabolic orbit about the sun collides with a body of equal mass at rest. Show that the center of mass of the two describes a circle about the sun as center.

6. Prove that the least velocity with which a body must be projected from the north pole so as to hit the surface of the earth at the equator is about $4\frac{1}{2}$ miles per second, and that the angle of elevation is 22°.5.

7. A particle moves in the common field of two fixed centers of force of equal intensity. The particle is attracted towards one of the centers with a force which varies as its distance from that center, and repelled from the other center according to the same law. Show that the orbit is a parabola.

8. A particle moves in a field in which the force is repulsive and varies inversely as the square of the distance from the center of force. Show that the orbit is a hyperbola.

9. In the preceding problem show that the radius vector sweeps over equal areas in equal intervals of time.

CHAPTER XV.

PERIODIC MOTION.

226. Simple Harmonic Motion. — When a particle moves in a straight line under the action of a force which is directed towards a fixed point and the magnitude of which varies directly as the distance of the particle from the fixed point, the motion is said to be *simple harmonic*.

Let O, Fig. 130, be the fixed point, m, the mass of the particle, and x its distance from O; then the foregoing definition gives

$$F = -kx, \tag{I'}$$

FIG. 130.

where k is the constant of proportionality. The negative sign in the right-hand member of the equation (I′) accounts for the fact that F is directed towards the fixed point, while x is measured in the opposite direction. Substituting this expression for F in the force equation we get

$$m\frac{dv}{dt} = -kx, \tag{I}$$

or

$$\frac{dv}{dt} = -\omega^2 x, \tag{I''}$$

where $\omega^2 = \dfrac{k}{m}$. Substituting $v\dfrac{dv}{dx}$ for $\dfrac{dv}{dt}$ in equation (I″) and integrating we have

$$v^2 = c^2 - \omega^2 x^2.$$

Let $v = v_0$ when $x = 0$, then $c = v_0$. Therefore

$$v = \sqrt{v_0^2 - \omega^2 x^2}. \tag{II}$$

Putting equation (II) in the form

$$\frac{dx}{dt} = \omega\sqrt{\frac{v_0^2}{\omega^2} - x^2}$$

and integrating we obtain

$$\sin^{-1}\frac{\omega x}{v_0} = \omega t + \delta,$$

or

$$x = \frac{v_0}{\omega}\sin(\omega t + \delta)$$

$$= a \sin(\omega t + \delta), \qquad \text{(III)}$$

where δ is the constant of integration and $a = \frac{v_0}{\omega}$.

227. Displacement. — The distance, x, of the particle from the fixed point is called the displacement.

228. Amplitude. — The maximum displacement is called the amplitude. It is evident from equation (III) that the amplitude equals a.

229. Phase. — The particle is said to be in the same phase at two different instants, if the displacement and the velocity at the one instant equal, respectively, the displacement and the velocity at the other instant.

230. Period. — The time which elapses between two successive instants at which the particle is in the same phase is called the period of the motion. In order to find the period we will make use of the definition of a periodic function.* It is evident from equation (III) that x is a periodic function of t; therefore we can write

$$x = a \sin[\omega t + \delta]$$
$$= a \sin[\omega(t + P) + \delta].$$

* If any variable x is a periodic function of any other variable t and if the dependence of x on t is given by the relation $x = f(t)$, then the function satisfies the following condition:

$$f(t) = f(t + nP),$$

where P is the period and n any positive or negative integer. As an illustration consider the function $x = \sin\theta$. This function evidently satisfies the relation $\sin\theta = \sin(\theta + n \cdot 2\pi)$. Therefore 2π is the period.

But since $\sin\theta$ is a periodic function of θ with a period of 2π, we have

$$\sin\theta = \sin(\theta + 2\pi),$$

therefore
$$x = a\sin[\omega(t+P)+\delta]$$
$$= a\sin[\omega t + \delta + 2\pi]$$

and consequently
$$P = \frac{2\pi}{\omega}. \qquad \text{(IV)}$$

231. Frequency. — The number of complete vibrations which the particle makes per second is called the frequency of the vibration. If n denotes the frequency, then

$$n = \frac{1}{P} = \frac{\omega}{2\pi}. \qquad \text{(V)}$$

232. Time-distance Diagram. — Suppose the particle to describe the vertical line AA', Fig. 131, the middle point of

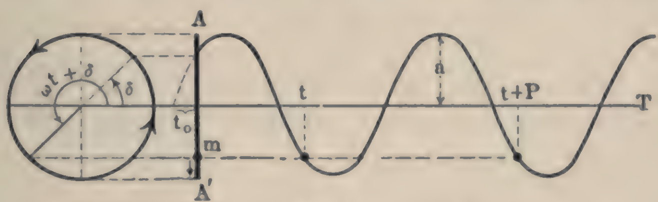


FIG. 131.

which is the fixed point. Then $OA = OA' = a$ = the amplitude. The relation between the position of the particle and the time may be visualized by plotting equation (III) with x as ordinate and t as abscissa. This gives the well-known sine curve.

A mental picture of the motion of the particle may be formed by supposing that the particle under consideration is a projection of another particle which moves in a circle of radius a with a constant speed. The second particle and its path may be called the *auxiliary particle* and the *auxiliary circle*, respectively.

233. Common Forms of Equation (III). — The following are the typical forms in which equation (III) is written:

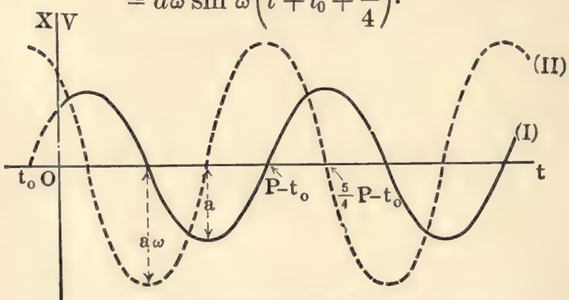
$$\left.\begin{aligned} x &= a \sin (\omega t + \delta) = a \sin \omega\, (t + t_0) \\ &= a \sin\left(\frac{2\pi}{P} t + \delta\right) = a \sin \frac{2\pi}{P} (t + t_0) \\ &= a \cos (\omega t + \delta') = a \cos \omega\, (t + t_0') \\ &= a \cos\left(\frac{2\pi}{P} t + \delta'\right) = a \cos \frac{2\pi}{P} (t + t_0'), \end{aligned}\right\} \qquad \text{(III}')$$

where $\delta' = \frac{\pi}{2} - \delta$ and $t_0' = \frac{P}{4} - t_0$.

234. Epoch. — The constants t_0 and t_0' are called *epochs*, and δ and δ' are called *epoch angles*. The meanings of these constants will be seen from Fig. 131.

235. Velocity. — The following expressions for the velocity of the particle may be obtained either from equation (II) or from equation (III):

$$\begin{aligned} v &= \sqrt{v_0{}^2 - \omega^2 x^2} = \omega \sqrt{a^2 - x^2} \\ &= a\omega \cos (\omega t + \delta) \\ &= a\omega \sin \left(\omega t + \delta + \frac{\pi}{2}\right) \\ &= a\omega \sin \omega \left(t + t_0 + \frac{P}{4}\right). \end{aligned}$$



(I) is the displacement – time curve

(II) is the velocity – time curve

FIG. 132.

It is evident from these expressions that the velocity is a simple harmonic function of the time, that it has the same

period as the displacement, and that it differs in phase from the latter by $\frac{P}{4}$, as shown in Fig. 132.

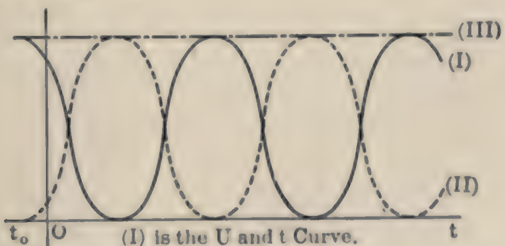
236. Energy of the Particle. — The following do not need further explanation.

$$
\begin{aligned}
U &= -\int_0^x F\,dx \\
&= \tfrac{1}{2}kx^2 \\
&= \frac{2\pi^2 m}{P^2}x^2 \\
&= \frac{2\pi^2 a^2 m}{P^2}\sin^2\frac{2\pi}{P}(t+t_0). \quad \text{(VII)}
\end{aligned}
\qquad
\begin{aligned}
T &= \tfrac{1}{2}mv^2 \\
&= \frac{2\pi^2 m}{P^2}(a^2-x^2) \\
&= \frac{2\pi^2 a^2 m}{P^2}\cos^2\frac{2\pi}{P}(t+t_0). \quad \text{(VI)}
\end{aligned}
$$

$$E = T + U = \frac{2\pi^2 a^2 m}{P^2}. \qquad \text{(VIII)}$$

Thus the total energy of the particle is constant and equals the maximum values of the potential and kinetic energies. The total energy varies, evidently, directly as the square of the amplitude and inversely as the square of the period.

In Fig. 133, T, U, and V are plotted as ordinates and the time as abscissa, with phase relations which correspond to the curves of Fig. 132.



(I) is the U and t Curve.
(II) is the T and t Curve.
(III) is the E and t Curve.

FIG. 133.

237. Average Value of the Potential Energy. — Since U may be considered as a function of either x or t, we will find its average value with respect to both variables. Taking 0 and

a as the limits of x and the corresponding values of t as the limits of t we have

$$\overline{U}_t = \frac{1}{\frac{P}{4} - 0}\int_0^{\frac{P}{4}} U\,dt^*$$
$$= \frac{4}{P}\,\frac{2\pi^2 a^2 m}{P^2}\int_0^{\frac{P}{4}} \sin^2 \frac{2\pi}{P} t\,dt$$
$$= \frac{4E}{P}\cdot\frac{P}{2\pi}\left[\frac{\pi}{P}t - \frac{1}{4}\sin\frac{4\pi t}{P}\right]_0^{\frac{P}{4}}$$
$$= \tfrac{1}{2}E.$$

$$\overline{U}_x = \frac{1}{a-0}\int_0^a U\,dx$$
$$= \frac{k}{2a}\int_0^a x^2\,dx$$
$$= \frac{ka^2}{6}$$
$$= \tfrac{1}{3}E.$$

$$\therefore \quad E = 3\,\overline{U}_x = 2\,\overline{U}_t. \qquad \text{(IX)}$$

PROBLEMS.

1. A particle which describes a simple harmonic motion has a period of 5 sec. and an amplitude of 30 cm. Find its maximum velocity and its maximum acceleration.

2. When a load of mass m is suspended from a helical spring of length L and of negligible mass an extension equal to D is produced. The load is pulled down through a distance a from its position of equilibrium and then set free. Find the period and the amplitude of the vibration. Hooke's law holds true.

3. Within the earth the gravitational attraction varies as the distance from the center. Suppose there were a straight shaft from pole to pole, with no resisting medium in it. What would be the period of oscillation of a body dropped into the shaft? Suppose the earth to be a sphere with a radius of 4000 miles.

4. In the preceding problem find the velocity with which the body would pass the center of the earth.

5. A particle describes a circle with constant speed. Show that the projection of the particle upon a straight line describes a simple harmonic motion.

6. The pan of a helical spring balance is lowered 2 inches when a weight of 5 pounds is placed on it. Find the period of vibration of the balance with the weight on.

* See footnote p. 142.

7. A particle which is constrained to move in a straight line is attracted by another particle fixed at a point outside the line. Show that the motion of the particle is simple harmonic when the force varies as the distance between the particles.

8. A particle of mass m describes a motion defined by the equation

$$x = a \sin (\omega t + \delta).$$

Find the average value of the following quantities, with respect to the time, for an interval of half the period:

(a) displacement;
(b) velocity;
(c) acceleration;
(d) force;
(e) momentum;
(f) kinetic energy;
(g) potential energy.

9. In problem 8 take the averages with respect to position.

10. In problem 8 suppose the motion to be given by

$$x = a \cos \omega (t + t_0).$$

11. In problem 10 take the averages with respect to position.

12. In problem 8 suppose the motion to be given by the following equations:

I. $x = a \sin^2 (\omega t + \delta)$.
II. $x = a \cos^2 (\omega t + \delta)$.
III. $x = a \sin \omega t \cos (\omega t + \delta)$.

238. Composition of Two Parallel Simple Harmonic Motions of Equal Period. Analytical Method. — Suppose

$$x_1 = a_1 \sin (\omega t + \delta_1), \tag{1}$$

and

$$x_2 = a_2 \sin (\omega t + \delta_2), \tag{2}$$

to define the motions which a particle would have if acted upon, separately, by two simple harmonic forces. Then the motion which will result when the forces act simultaneously is obtained by adding equations (1) and (2). Thus

$$\begin{aligned} x &= x_1 + x_2 \\ &= a_1 \sin (\omega t + \delta_1) + a_2 \sin (\omega t + \delta_2). \end{aligned}$$

Expanding the right-hand member of the last equation and rearranging the terms we get

$$\begin{aligned} x &= (a_1 \cos \delta_1 + a_2 \cos \delta_2) \sin \omega t \\ &\qquad + (a_1 \sin \delta_1 + a_2 \sin \delta_2) \cos \omega t \\ &= a \cos \delta \sin \omega t + a \sin \delta \cos \omega t \\ &= a \sin (\omega t + \delta), \end{aligned} \tag{3}$$

where $\qquad a \cos \delta = a_1 \cos \delta_1 + a_2 \cos \delta_2,$

and $\qquad a \sin \delta = a_1 \sin \delta_1 + a_2 \sin \delta_2.$

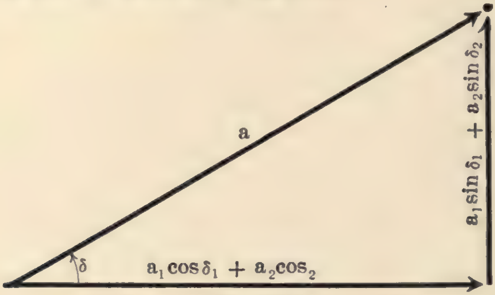


FIG. 134.

It is evident from equation (3) that the resulting motion is simple harmonic and has the same period as the component motions.

Squaring the last two equations and adding we obtain the amplitude of the motion in terms of the constants of equations (1) and (2). Thus

$$a^2 = a_1^2 + a_2^2 + 2\, a_1 a_2 \cos (\delta_2 - \delta_1). \tag{4}$$

The phase angle of the motion is evidently defined by

$$\tan \delta = \frac{a_1 \sin \delta_1 + a_2 \sin \delta_2}{a_1 \cos \delta_1 + a_2 \cos \delta_2}. \tag{5}$$

239. Graphical Method. — The graph of the resulting motion may be obtained by either of the following methods:

(1) Represent the given motions by displacement-time curves, then add the ordinates of these curves in order to

obtain the curve which represents the resultant motion. In Fig. 135 the curves (I) and (II) represent the component motions and curve (III) represents the resultant motion.

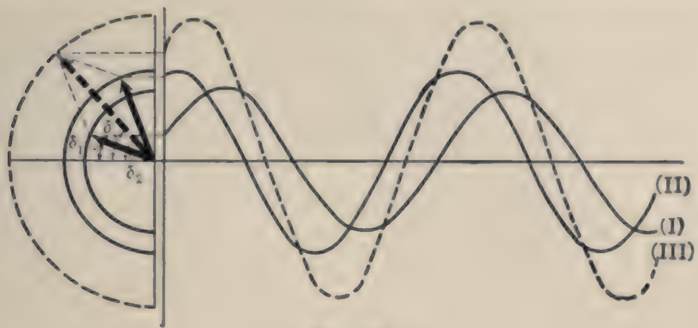


FIG. 135.

(2) Draw two concentric auxiliary circles with radii equal to the amplitudes of the component motions; draw a radius in each circle making an angle with the t-axis, equal to the phase angle of the corresponding motion; the vector sum of these radii gives the radius of the auxiliary circle for the resultant motion and the corresponding phase angle. By the help of this auxiliary circle the displacement-time curve of the resulting motion can be drawn without drawing those of the component motions.

PROBLEMS.

Find the resultant motion due to the superposition of two motions defined by the following pairs of equations:

(1) $x_1 = a_1 \sin \omega t$ and $x_2 = a_2 \sin\left(\omega t + \frac{\pi}{2}\right)$.

(2) $x_1 = a_1 \sin \omega t$ and $x_2 = a_2 \cos\left(\omega t - \frac{\pi}{4}\right)$.

(3) $x_1 = a_1 \cos \omega t$ and $x_2 = a_2 \cos\left(\omega t + \frac{\pi}{3}\right)$.

(4) $x_1 = a_1 \sin \omega t$ and $x_2 = a_2 \sin (\omega t + \delta)$.

(5) $x_1 = a_1 \sin \omega t$ and $x_2 = a_2 \cos \omega t$.

(6) $x_1 = a_1 \sin \frac{2\pi}{P} t$ and $x_2 = a_2 \cos \frac{2\pi}{P} (t + t_0)$.

(7) $x_1 = a_1 \cos \omega t$ and $x_2 = a_2 \sin (\omega t + \delta)$.

(8) $x_1 = a_1 \cos \omega t$ and $x_2 = a_2 \cos (\omega t + \delta)$.

(9) $x_1 = a_1 \cos \omega t$ and $x_2 = a_2 \cos \frac{2\pi}{P} (t + t_0)$.

(10) $x_1 = a_1 \sin \left(\omega t + \frac{\pi}{2}\right)$ and $x_2 = a_2 \sin \frac{2\pi}{P} t$.

(11) $x_1 = a_1 \sin (\omega t + \delta_1)$ and $x_2 = a_2 \cos (\omega t + \delta_2)$.

(12) $x_1 = a_1 \cos (\omega t + \delta_1)$ and $x_2 = a_2 \cos (\omega t + \delta_2)$.

(13) $x_1 = a_1 \cos \frac{2\pi}{P} (t + t_0)$ and $x_2 = a_2 \sin \frac{2\pi}{P} (t - t_0)$.

(14) $x_1 = a_1 \sin \frac{2\pi}{P} (t - t_0)$ and $x_2 = a_2 \sin \frac{2\pi}{P} (t + t_0)$.

(15) $x_1 = a_1 \cos \frac{2\pi}{P} (t - t_0)$ and $x_2 = a_2 \cos \frac{2\pi}{P} (t + t_0)$.

240. Elliptic Harmonic Motion. — Consider the motion of a particle which is acted upon by two harmonic forces whose directions are perpendicular to each other. Suppose the periods of vibration of the particle due to the separate action of the forces to be the same, then the following equations define the component motions.

$$x = a \sin \omega t,^* \tag{1}$$

$$y = b \sin (\omega t + \delta). \tag{2}$$

The equation of the path of the particle may be obtained by eliminating t between equations (1) and (2). Expanding the right-hand member of equation (2) and substituting for $\sin \omega t$ and $\cos \omega t$ from equation (1) we get

$$\begin{aligned} y &= b \sin \omega t \cdot \cos \delta + b \cos \omega t \cdot \sin \delta \\ &= b \frac{x}{a} \cos \delta + b \sqrt{1 - \frac{x^2}{a^2}} \cdot \sin \delta, \end{aligned}$$

* The phase angle is left out of equation (1) to simplify the problem. This, however, does not affect the generality of the problem. It simply amounts to choosing a particular instant as the origin of the time axis. If, however, the phase angle is left out of both of the component motions the generality of the problem is affected because that will amount to assuming that the component motions are in the same phase.

or $$\frac{y}{b} - \frac{x}{a}\cos\delta = \sqrt{1 - \frac{x^2}{a^2}} \cdot \sin\delta.$$

Squaring the last equation and simplifying we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2\,xy}{ab}\cos\delta = \sin^2\delta, \tag{3}$$

which is the equation of an ellipse, Fig. 136. The following cases are of special interest.

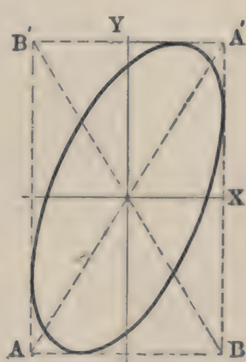


FIG. 136.

Case I. When $\delta = 0$, equation (3) reduces to $y = \frac{b}{a}x$, which is the equation of the line AA'. Substituting the values of x and y in the equation

$$r = \sqrt{x^2 + y^2}$$

we obtain

$$r = \sqrt{a^2 + b^2} \cdot \sin\omega t$$

for the equation of the motion. Therefore the motion is simple harmonic, in the line AA', with an amplitude equal to $\sqrt{a^2 + b^2}$ and period $\frac{2\pi}{\omega}$.

Case II. When $\delta = \pi$, equation (3) reduces to $y = -\frac{b}{a}x$. Therefore the motion is similar to that in Case I and takes place in the line BB'.

Case III. When $\delta = \pm\frac{\pi}{2}$ equation (3) reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

while equations (1) and (2) become

$$x = a\sin\omega t,$$
$$y = b\cos\omega t.$$

In this case, therefore, the particle describes an elliptical

path with a period equal to $\frac{2\pi}{\omega}$. The axes of the path coincide with the coördinate axes.

Case IV. When $\delta = \pm\frac{\pi}{2}$ and $b = a$, the path becomes a circle, and the motion uniform circular motion with a period equal to $\frac{2\pi}{\omega}$.

PROBLEMS.

Find the resultant motion due to the superposition of the motions defined by the following equations:

(1) $x = a\cos\omega t$ and $y = a\sin\omega t$.

(2) $x = a\cos\frac{2\pi}{P}(t + t_0)$ and $y = a\sin\frac{2\pi}{P}t$.

(3) $x = a\sin\omega t$ and $y = a\cos\frac{2\pi}{P}(t + t_0)$.

(4) $x = a\sin(\omega t + \delta)$ and $y = a\cos(\omega t - \delta)$.

(5) $x = a\cos\omega t$ and $y = b\sin\omega t$.

(6) $x = a\sin(\omega t - \delta)$ and $y = b\cos(\omega t + \delta)$.

(7) $x = a\sin\left(\omega t - \frac{\pi}{2}\right)$ and $y = b\sin\left(\omega t + \frac{\pi}{2}\right)$.

(8) $x = a\cos\left(\omega t + \frac{\pi}{3}\right)$ and $y = b\sin\left(\omega t - \frac{\pi}{3}\right)$.

(9) $x = a\cos\left(\omega t - \frac{\pi}{4}\right)$ and $y = b\cos\left(\omega t + \frac{\pi}{4}\right)$.

(10) $x = a\cos(\omega t + \delta_1)$ and $y = b\sin(\omega t + \delta_2)$.

241. Physical Pendulum. — Any rigid body which is free to oscillate under the action of its own weight is called a *physical* or a *compound pendulum.* Let A, Fig. 137, be a rigid body which is free to oscillate about a horizontal axis through the point O and perpendicular to the plane of the paper. Further let c denote the position of the center of mass and D its distance from the axis. Then the torque equation gives

$$I\frac{d\omega}{dt} = -mgD\sin\theta, \qquad \text{(X)}$$

where m is the mass of the body and θ the angular displacement from the position of equilibrium.

The equation $\frac{d^2x}{dt^2} = -k^2 \sin x$ is not integrable in a finite number of terms; therefore the solution of equation (X) must be given either in an approximate form, or it must be expressed as an infinite series.

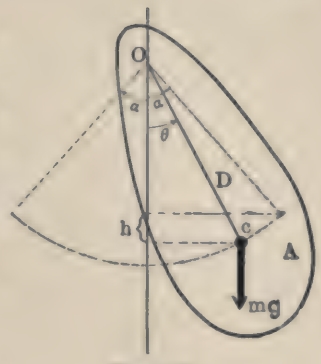


FIG. 137.

FIRST APPROXIMATION. — When θ is small $\sin\theta$ may be replaced by θ. Therefore we can write

$$I\frac{d\omega}{dt} = -mgD\theta, \tag{X'}$$

or

$$\frac{d\omega}{dt} = -c^2\theta, \tag{X''}$$

where $c^2 = \frac{mgD}{I}$. It will be observed that the last two equations are of the same type as equations (I′) and (II′) of p. 297, the differential equations of simple harmonic motion. Therefore the motion of the physical pendulum is approximately harmonic. Hence we can apply to the present problem the results which were obtained in discussing simple harmonic motion. Thus the expression for the displacement is

$$\theta = \alpha \sin(\omega t + \delta), \tag{XI}$$

where α is the amplitude, i.e., the maximum angular displacement of the pendulum. On the other hand the period of the pendulum is

$$P_0 = \frac{2\pi}{c}$$

$$= 2\pi\sqrt{\frac{I}{mgD}} \tag{XII}$$

$$= 2\pi\sqrt{\frac{I_c + mD^2}{mgD}}$$

$$= 2\pi\sqrt{\frac{K^2 + D^2}{gD}}, \tag{XIII}$$

where I_c is the moment of inertia of the pendulum about an axis through the center of mass parallel to the axis of vibration and K is the corresponding radius of gyration.

Second Approximation. — Starting with the energy equation we have

$$\frac{1}{2} I \left(\frac{d\theta}{dt}\right)^2 = mgh$$

$$= mgD(\cos\theta - \cos\alpha),$$

or
$$dt = \sqrt{\frac{I}{2\,mgD}} \cdot \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}$$

$$= \frac{1}{2}\sqrt{\frac{I}{mgD}} \cdot \frac{d\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}} \cdot \quad \left[\cos\theta = 1 - 2\sin^2\frac{\theta}{2}\right]$$

Integrating the left-hand member of the last equation between the limits $t = 0$ and $t = \frac{P}{4}$, and indicating the integration of the right-hand member between the corresponding limits we have

$$\frac{P}{4} = \frac{1}{2}\sqrt{\frac{I}{mgD}} \int_0^\alpha \frac{d\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}}.$$

The last integral cannot be evaluated in a finite number of terms, but we can expand the integrand into a power series, every term of which is integrable.

Let
$$\sin\frac{\theta}{2} = \sin\frac{\alpha}{2}\sin\phi.$$

Then $\phi = 0$ when $\theta = 0$, and $\phi = \frac{\pi}{2}$ when $\theta = \alpha$; further

$$d\theta = \frac{2\sin\frac{\alpha}{2}\cos\phi \cdot d\phi}{\sqrt{1 - \sin^2\frac{\alpha}{2}\sin^2\phi}}.$$

Making these substitutions in the left-hand member of the preceding expression for the period we obtain

$$P = 4\sqrt{\frac{I}{mgD}}\int_0^{\frac{\pi}{2}}\left(1 - \sin^2\frac{\alpha}{2}\sin^2\phi\right)^{-\frac{1}{2}}d\phi\ *$$

$$= 4\sqrt{\frac{I}{mgD}}\int_0^{\frac{\pi}{2}}\left(1 + \frac{1}{2}\sin^2\frac{\alpha}{2}\sin^2\phi + \frac{1\cdot 3}{1\cdot 2\cdot 3}\sin^4\frac{\alpha}{2}\sin^4\phi + \cdots\right)d\phi\ \dagger$$

$$= 4\sqrt{\frac{I}{mgD}}\int_0^{\frac{\pi}{2}}\left[1 + \frac{1}{4}\sin^2\frac{\alpha}{2}(1 - \cos 2\phi) + \cdots\right]d\phi$$

$$= 4\sqrt{\frac{I}{mgD}}\left(\frac{\pi}{2} + \frac{\pi}{8}\sin^2\frac{\alpha}{2} + \cdots\right)$$

$$= P_0\left(1 + \frac{1}{4}\sin^2\frac{\alpha}{2} + \cdots\right) \quad \left[\text{by (XII)}\right]$$

$$\doteq P_0\left(1 + \frac{1}{4}\sin^2\alpha\right) \quad \left[\begin{matrix}\text{when } \alpha \text{ is small higher}\\ \text{terms may be neglected}\end{matrix}\right]$$

$$\doteq P_0\left(1 + \frac{\alpha^2}{16}\right) \quad \left[\text{when } \alpha \text{ is small } \sin\frac{\alpha}{2} \doteq \frac{\alpha}{2}\right].$$

Therefore

$$P_0 = P\left(1 - \frac{\alpha^2}{16}\right).\ \ddagger \qquad \text{(XIV)}$$

242. Simple Pendulum. — A ball which is suspended by means of a string forms a simple pendulum when it is free to swing about a horizontal axis through the upper end of the string, provided the mass of the string is negligible compared with that of the ball and the radius of the ball is negligible compared with the length of the string. If m denotes the mass of the ball and l the length of the string then

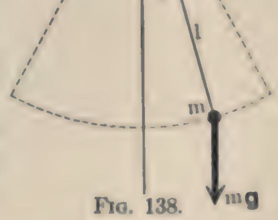


Fig. 138.

* This is called an elliptic integral.

† This expansion is carried out by the Binomial theorem. See Appendix A.

‡ See Appendix A₁.

the moment of inertia of the pendulum equals ml^2. Therefore substituting this value of I in the expressions for P_0 and P and replacing D by l we obtain

$$P_0 = 2\pi\sqrt{\frac{I}{mgD}}$$

$$= 2\pi\sqrt{\frac{l}{g}}, \qquad \text{(XV)}$$

for the first approximation, and

$$P = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\alpha^2}{16}\right),$$

$$= P_0\left(1 + \frac{\alpha^2}{16}\right) \qquad \text{(XV}')$$

for the second approximation.

243. Equivalent Simple Pendulum. — A simple pendulum which has the same period as a physical pendulum is called the *equivalent simple pendulum* of the latter. If l denotes the length of the equivalent simple pendulum, then

$$P = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{K^2 + D^2}{gD}}.$$

$$\therefore \quad l = \frac{K^2 + D^2}{D}. \qquad \text{(XVI)}$$

For a given value of D and a given direction of the axis, K is constant. Therefore if the direction of the axis is not changed l is a function of D alone. If we plot the last equation with l as ordinate and D as abscissa we obtain a curve similar to that of Fig. 139. It is evident from the curve that the value of l is infinitely large for $D = 0$, but it diminishes rapidly to the minimum value $\frac{2K}{g}$ as D reaches the value K. As D is increased further l increases continually. It will be observed that for a given value of l greater than $\frac{2K}{g}$ there are two values of D, one of which is less and the other greater than K.

The group of parallel axes about which the rigid body oscillates with the same period forms two coaxial circular cylinders, Fig. 140, whose common axis passes through the cen-

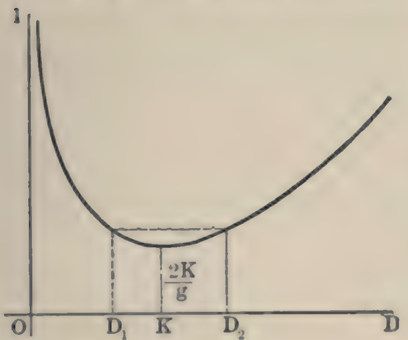


FIG. 139.

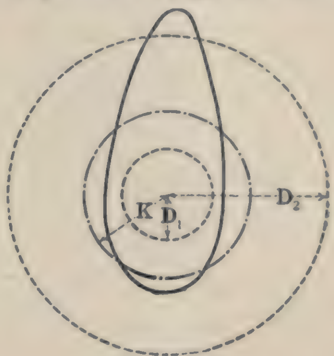


FIG. 140.

ter of mass. The cylinders which correspond to the minimum value of the period coincide and have a common radius K.

PROBLEMS.

1. Find the period of the following physical pendulums:

(a) A uniform rod, the transverse dimensions of which are negligible compared with the length, oscillates about a horizontal axis through one end.

(b) A sphere suspended from a horizontal axis by means of a string of negligible mass. Discuss the changes in the period as the axis approaches the center of the sphere.

(c) A circular flat ring oscillates about an axis which forms an element of the inner surface.

(d) A door oscillates about the line of the hinges which make an angle α with the vertical.

2. A sphere of radius a oscillates back and forth in a perfectly smooth spherical bowl of radius b. Find the period of oscillation. The sphere is supposed to have no rolling motion.

3. What effect on the period of a pendulum would be produced by a change in the mass of the bob, or of the length of the string, or in the radius of the earth, or in the length of the day, or in the latitude of the location?

4. A seconds pendulum loses 30 seconds per day at the summit of a mountain. Find the height of the mountain, considering the earth to be

a sphere of 4000 miles radius and the gravitational force to vary inversely as the square of the distance from the center of the earth.

5. Given the height of a mountain above the surrounding plain and the period of a pendulum on the plain and on the top of the mountain, find a relation from which the radius of the earth can be computed.

6. Supposing the gravitational attraction within the earth to vary as the distance from the center, find the depth below the surface at which a seconds pendulum will beat 2 seconds.

7. Derive a relation between the distance of a pendulum from the center of the earth and its period.

8. A balloon ascends with a constant acceleration and reaches 400 feet in one minute. What is the rate at which the pendulum gains in the balloon?

9. A pendulum of length l is shortened by a small amount δl. Show that it will gain about $\frac{n \cdot \delta l}{2\,l}$ vibrations in an interval of time of n vibrations. n is supposed to be a large integral number.

10. How high above the surface of the earth must a seconds pendulum be carried in order that it may have a period of 4 seconds?

11. While a train is taking a curve at the rate of 60 miles per hour a seconds pendulum hanging in the train is observed to swing at the rate of 121 oscillations in 2 minutes. Show that the radius of the curve is about a quarter of a mile.

12. Find the expressions for the least period of oscillation the following bodies can have; also determine the corresponding position of the axes.

(a) Rod of negligible transverse dimensions.
(b) Square plate of negligible thickness.
(c) Circular plate of negligible thickness.
(d) Solid cylinder.
(e) Solid sphere.
(f) Spherical shell.

244. Determination of the Gravitational Acceleration by Means of a Reversible Pendulum. — A physical pendulum which is provided with two convenient axes of vibration is called a reversible pendulum. Let D and D', Fig. 141, denote the distances of the axes from the center of mass. Then the corresponding periods are

$$P = 2\,\pi\sqrt{\frac{K^2 + D^2}{gD}}$$

and

$$P' = 2\,\pi\sqrt{\frac{K^2 + D'^2}{gD'}}.$$

Eliminating K and solving for g we get

$$g = 4\pi^2 \frac{D'^2 - D^2}{D'P'^2 - DP^2}. \tag{1}$$

Reversible pendulums which are made for the purpose of determining g are so constructed that the two periods are very nearly equal. Therefore we can write

$$P' = P + \delta P, \quad [\delta P \ll P],$$

and obtain

$$\begin{aligned} g &= 4\pi^2 \frac{D'^2 - D^2}{D'(P+\delta P)^2 - DP^2} \\ &= 4\pi^2 \frac{D'^2 - D^2}{P^2(D'-D) + 2PD'\delta P + D'(\delta P)^2} \\ &= 4\pi^2 \frac{D + D'}{P^2\left(1 + \dfrac{2D'}{D'-D}\cdot\dfrac{\delta P}{P}\right)} \qquad [(\delta p)^2 \text{ is neglected}] \\ &= 4\pi^2 \frac{D+D'}{P^2}\left(1 - \frac{2D'}{D'-D}\cdot\frac{\delta P}{P}\right).^* \end{aligned} \tag{2}$$

The approximate expression which is given in equation (2) is better adapted for computing the value of g from experimental data than the more exact expression given in equation (1). This is due to the fact that $(D' - D)$, which cannot be determined with a high degree of accuracy, enters into equation (1) as a factor, while it appears only in the correction term of equation (2).

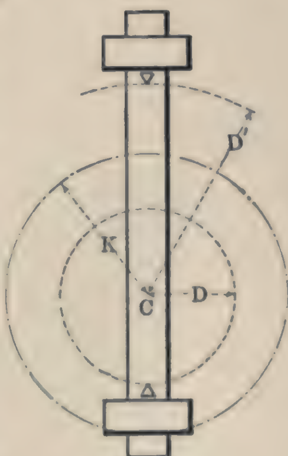


Fig. 141.

245. Bifilar Pendulum. — A rigid body which is suspended by means of two parallel strings, as shown in Fig. 142, is called a *bifilar pendulum*. When the body is given an angular displacement about a vertical axis through

* See Appendix A1.

its center of mass and then left to itself it vibrates with a definite period.

Let $m =$ the mass of the rigid body,
$I =$ the moment of inertia of the body about a vertical axis through its center of mass,
$l =$ the length of each string,
$\theta =$ the angular displacement of the bar,
$\phi =$ the angular displacement of the strings,
$T =$ the tensile forces of the strings, and
$2\,D =$ the distance between the strings.

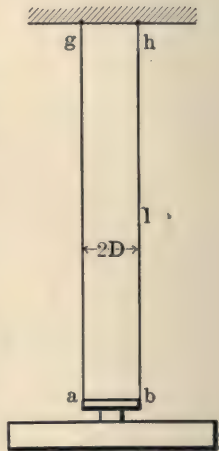


FIG. 142.

In order to obtain the torque equation suppose the weight of the suspended system to be concentrated at the ends of the small bar ab and analyze the forces acting upon it as shown in Fig. 143. Evidently ab is acted upon by four forces, namely, the tensile forces of the strings and the two forces each of which represents half the weight of the suspended system. These forces are equivalent to a couple formed by the forces $\mathbf{F}$ and $-\mathbf{F}$, which act at the ends of ab in a horizontal direction, and a vertical force equal to the difference between the sum of the vertical components of the tensile forces of the strings and the weight of the suspended system. The vertical force gives the suspended system a motion in the vertical direction. But both this motion and the force which produces it are very small therefore they will be neglected.

It is evident from Fig. 143 that the torque due to the horizontal couple is

$$G = -2 \cdot F \cdot D \cos\frac{\theta}{2}.$$

Substituting this value of G in the torque equation we have

$$I\frac{d\omega}{dt} = -2FD\cos\frac{\theta}{2}$$

$$= -2TD\sin\phi\cos\frac{\theta}{2}.$$

FIRST APPROXIMATION. — When θ and ϕ are small the following relations give close enough approximations.

FIG. 143.

$$T = \tfrac{1}{2}mg, \quad \cos\frac{\theta}{2} = 1,$$

$$D\theta = l\phi,^* \quad \sin\phi = \phi = \frac{D}{l}\theta.$$

Making these substitutions in the torque equation we get

$$I\frac{d\omega}{dt} = -\frac{mgD^2}{l}\theta,$$

which is the equation of simple harmonic motion. Therefore

$$P_0 = \frac{2\pi}{D}\sqrt{\frac{Il}{mg}}$$

is the period of the motion.

SECOND APPROXIMATION. — From Fig. 143 we have

$$T\cos\phi = \tfrac{1}{2}mg \quad \text{and} \quad ea' = l\sin\phi = 2D\sin\frac{\theta}{2}.$$

Therefore $$\sin\phi = \frac{2D}{l}\sin\frac{\theta}{2}$$

and $$T = \frac{mg}{2\cos\phi} = \frac{mg}{2\sqrt{1 - \frac{4D^2}{l^2}\sin^2\frac{\theta}{2}}}.$$

* The line ea' is considered as an arc of each of two circles with centers at g and c.

Making these substitutions in the torque equation we obtain

$$I\frac{d\omega}{dt} = -\frac{mgD^2}{l}\cdot\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\sqrt{1-\frac{4D^2}{l^2}\sin^2\frac{\theta}{2}}}$$

$$= -\frac{mgD^2}{l}\frac{\sin\theta}{\sqrt{1-\frac{4D^2}{l^2}\sin^2\frac{\theta}{2}}}.$$

In actual experiments $\frac{D}{l}$ is made less than $\frac{1}{20}$. Therefore even if the maximum value of θ is made as large as half a radian the second term under the radical is less than $\frac{1}{1600}$ and consequently negligible. Thus the last equation reduces to

$$I\frac{d\omega}{dt} = -\frac{mgD^2}{l}\sin\theta,$$

which is the well-known pendulum equation. Therefore we have

$$P = \frac{2\pi}{D}\sqrt{\frac{Il}{mg}}\left(1+\frac{\alpha^2}{16}\right)$$

$$= P_0\left(1+\frac{\alpha^2}{16}\right)$$

for a second approximation to the actual value of the period.

FIG. 144.

246. Torsional Pendulum. — A torsional pendulum consists of a rigid body suspended by a wire, the wire being rigidly connected to both the support and the body, Fig. 144. When the body is given an angular displacement about the wire as an axis and then left to itself it vibrates with a constant period. The torque which produces the angular displacement obeys Hooke's law; therefore

$$G = -k\theta,$$

where k is a positive constant which depends upon the physical properties of the wire.* The negative sign indicates the fact that the torque and the angular displacement are oppositely directed. Substituting this value of G in the torque equation we have

$$I\frac{d\omega}{dt} = -k\theta, \tag{XVII}$$

or

$$\frac{d\omega}{dt} = -c^2\theta,$$

where $c^2 = \frac{k}{I}\cdot$ But these are the typical forms of the equation of simple harmonic motion; therefore

$$P = \frac{2\pi}{c} = 2\pi\sqrt{\frac{I}{k}} \tag{XVIII}$$

is the expression for the period. It will be observed that the motion is strictly harmonic; consequently there is no correction for finite amplitudes.

247. Application to the Determination of Moment of Inertia. — Let P be the period of the torsion pendulum and P' its period after the body whose moment of inertia is desired is fastened to the bob of the pendulum. Further let I be the moment of inertia of the bob about the suspension wire as an axis and I' the moment of inertia of the body. Then we have

$$P = 2\pi\sqrt{\frac{I}{k}}$$

and

$$P' = 2\pi\sqrt{\frac{I+I'}{k}}.$$

Therefore

$$I' = \frac{P'^2 - P^2}{P^2}I$$

and

$$k = 4\pi^2\frac{I'}{P'^2 - P^2}.$$

Hence if I is known both I' and k may be determined experimentally.

* Page 178.

248. Damped Harmonic Motion. — When a particle moves in a harmonic field of force which is filled by a resisting medium the motion of the particle is called *damped harmonic* motion. The particle is acted upon by two forces, namely, a harmonic force due to the field, and a resisting force due to the medium. All resisting forces are functions of the velocity and act in a direction opposed to that of the velocity. But since in harmonic motion the velocity does not attain great values, we can suppose the resisting force to be a linear function of the velocity. Therefore if F denotes the total force acting upon the particle we can write

$$F = -k_1x - k_2v,$$

where the first term of the right-hand member represents the harmonic force and the second term the resisting force. Substituting this value of F in the force equation we get

$$m\frac{dv}{dt} = -k_1x - k_2v. \qquad \text{(XIX)}$$

A motion which is the perfect analogue of the motion defined by equation (XIX) is obtained when a rigid body placed in a resisting medium is subjected to a harmonic torque. The motion is defined by the following torque equation:

$$I\frac{d\omega}{dt} = -k'\theta - k''\omega, \qquad \text{(XX)}$$

where the first term of the right-hand member represents the harmonic torque and the second term the resisting torque.

On account of the perfect analogy between the two types of motion a discussion of one of them is all that is necessary. We will consider the motion represented by equation (XX). Let $\frac{k''}{I} = 2a$ and $\frac{k'}{I} = b^2$, then equation (XX) becomes

$$\frac{d^2\theta}{dt^2} + 2a\frac{d\theta}{dt} + b^2\theta = 0. \qquad (1)$$

The last equation is a differential equation of the second order which can be solved by the well-known methods of Differential Equations. We will, however, obtain the solution by a method which is more instructive and which may be called an experimental method.

It will be observed that θ and its first two derivatives are added in equation (1); therefore θ must be such a function of t that when it is differentiated with respect to the time the result is a function of the same type. The only known elementary functions which satisfy this condition are the circular and exponential functions. But since circular functions may be obtained from exponential functions* the solution of equation (1) may be expressed in the form

$$\theta = \alpha e^{\beta t}, \tag{2}$$

where α and β are constants. Replacing θ and its first two derivatives in equation (1) by their values, which are obtained from equation (2), we get

$$(\beta^2 + 2\,a\beta + b^2)\,\alpha e^{\beta t} = 0.$$

Evidently one or both of the factors must vanish. When $\alpha e^{\beta t} = 0$, $\theta = 0$, which means that there is no motion. This is called a trivial solution. When the other factor vanishes we get

$$\beta = -\,a \pm \sqrt{a^2 - b^2}.$$

Substituting these values of β in equation (2) we obtain the following particular solutions:

$$\theta' = \alpha e^{-(a + \sqrt{a^2 - b^2})t}.$$

$$\theta'' = \alpha e^{-(a - \sqrt{a^2 - b^2})t}.$$

In order to obtain the general solution we multiply the particular solutions by constants and add them. Hence

$$\theta = \alpha e^{-at}\left(c_1 e^{\sqrt{a^2 - b^2}\,t} + c_2 e^{-\sqrt{a^2 - b^2}\,t}\right)$$

* See Appendix Avii.

is the general solution of equation (2). Now let $\theta = 0$ when $t = 0$, then $c_2 = -c_1$. Therefore

$$\theta = A_1 e^{-at}\left(e^{\sqrt{a^2-b^2}\,t} - e^{-\sqrt{a^2-b^2}\,t}\right), \qquad \text{(XXI)}$$

where $A_1 = \alpha c_1$. There are three special cases which must be discussed separately.

Case I. Let $a^2 = b^2$, then $\theta = 0$ for all values of the time. Therefore this is a case of no motion.

Case II. Let $a^2 > b^2$, then $\sqrt{a^2 - b^2}$ is real. Denoting this radical by c we have

$$\theta = A_1 \left[e^{-(a-c)t} - e^{-(a+c)t}\right]. \qquad \text{(XXII)}$$

The character of the motion is brought out by the graph of equation (XXII), Fig. 145. The graph is easily obtained by drawing the dotted curves, which are plotted by considering the terms of the right-hand member of equation (XXII) separately, and then adding them geometrically. It is evident from the curve that the value of θ starts at zero, increases to a maximum, and then diminishes to zero asymptotically. In this case the motion is said to be *aperiodic* or *dead-beat*.

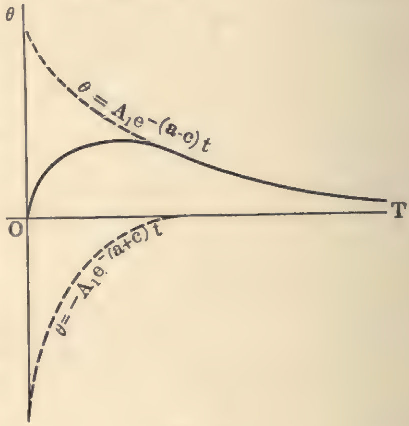


FIG. 145.

Case III. Let $a^2 < b^2$ then $\sqrt{a^2 - b^2}$ is imaginary. Let $\sqrt{-1} = i$ and $\sqrt{b^2 - a^2} = \omega$. Then $\sqrt{a^2 - b^2} = i\omega$. Making this substitution in equation (XXI) we obtain

$$\begin{aligned} \theta &= A_1 e^{-at}\left(e^{ikt} - e^{-ikt}\right) \\ &= A_1 e^{-at} \cdot 2\,i \sin \omega t\,* \\ &= A e^{-at} \sin \omega t, \qquad \text{(XXIII)} \end{aligned}$$

* See Appendix Avii.

where $A = 2\,iA_1$. Equation (XXIII) is the integral equation of harmonic motion with the additional factor e^{-at}, which is called the *damping factor*. On account of this factor the amplitude of the motion continually diminishes.

It is evident from equation (XXIII) that the motion is periodic and has a period

$$P = \frac{2\pi}{\omega}$$

$$= \frac{2\pi}{\sqrt{b^2 - a^2}}. \qquad \text{(XXIV)}$$

The character of the motion is brought out clearly by the displacement-time curve of Fig. 146. A mental picture of the

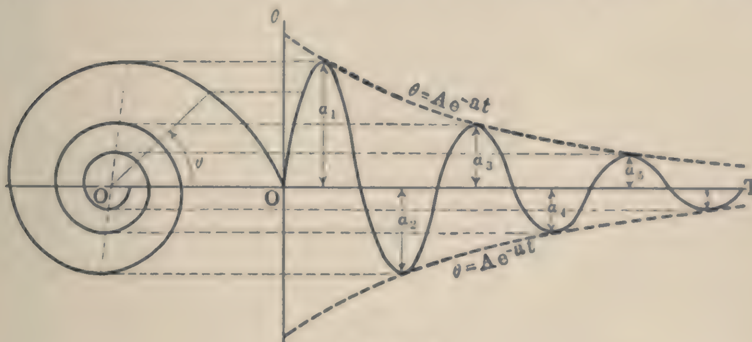


FIG. 146.

damped harmonic motion of a particle may be formed by considering the motion of an auxiliary particle which moves in a logarithmic spiral. If the auxiliary particle describes the logarithmic spiral of the figure in the counter-clockwise direction, in such a way as to give the radius vector a constant angular velocity, then the motion of the projection of the auxiliary particle upon the θ-axis is damped harmonic.

The logarithmic spiral may be used as an auxiliary curve in drawing the graph of equation (XXIII), as the circle is used in drawing a sine curve.

249. Logarithmic Decrement. — The logarithm of the ratio of two consecutive amplitudes is constant and is called the *logarithmic decrement* of the motion. The amplitudes occur whenever the relation

$$\tan(\omega t) = \frac{\omega}{a}^{*}$$

is satisfied. Let the first amplitude occur at the instant $t = t_1$; then since the period of the tangent is π, the times of the succeeding amplitudes are given by

$$\tan(\omega t) = \tan(\omega t_1 + n\pi),$$

or by

$$t = t_1 + \frac{n\pi}{\omega},$$

where n is a positive integer. Hence, denoting the logarithmic decrement by λ and the nth amplitude by α_n, we have

$$\begin{aligned}
\lambda &= \log\frac{\alpha_n}{\alpha_{n+2}} \qquad \text{(by definition)}\\
&= \log\frac{Ae^{-a\left(t_1+\frac{n\pi}{\omega}\right)}\sin(\omega t_1 + n\pi)}{Ae^{-a\left(t_1+\frac{n+2}{\omega}\pi\right)}\sin[\omega t_1 + (n+2)\pi]} \quad \text{[by (XXIII)]}\\
&= \log\frac{e^{-a\left(t_1+\frac{n\pi}{\omega}\right)}}{e^{-a\left(t_1+\frac{n+2}{\omega}\pi\right)}}\\
&= a\frac{2\pi}{\omega}\\
&= aP\\
&= \frac{k''}{2I}P. \qquad \text{(XXV)}
\end{aligned}$$

Therefore if I is known k'' may be determined from observations of P and α.

* Obtained by setting $\frac{d\theta}{dt} = 0$.

250. Effect of Damping on the Period. — Substituting the values of a and b in the expression for the period,

$$P = \frac{2\pi}{\sqrt{\frac{k'}{I} - \frac{\lambda^2}{P^2}}}\cdot$$

$$= 2\pi\sqrt{\frac{I}{k'}}\left(1 + \frac{\lambda^2}{4\pi^2}\right)^{\frac{1}{2}}$$

$$= 2\pi\sqrt{\frac{I}{k'}}\left(1 + \frac{\lambda^2}{8\pi^2} + \cdot\cdot\cdot\right)$$

$$\doteq P_0\left(1 + \frac{\lambda^2}{8\pi^2}\right), \qquad \text{(XXVI)}$$

where P_0 is the period for the undamped motion. It is evident from equation (XXVI) that the damping increases the period.

VIBRATIONS ABOUT A POSITION OF EQUILIBRIUM.

251. Lagrange's Method.— In the various pendulum problems which we have discussed the vibrating body was considered to be either a particle or a rigid body. These simplifications were necessary because the methods we have used cannot be applied conveniently to complicated systems. Lagrange (1736–1813) introduced into Dynamics a method which can be applied to any vibrating system. The following is a special case of his method adapted to conservative systems which have only one degree of freedom of motion.

Express the potential energy of the system as a function of a properly chosen* coördinate q, so that when expanded in ascending powers of q the first power of q does not appear. Then the potential energy takes the form

$$U = \beta_0 + \beta_2 q^2 + \beta_3 q^3 + \cdot\cdot\cdot, \qquad \text{(XXVII)}$$

where β_0, β_2, etc., are constants. The constant β_0 can be

* It is shown in books on advanced Dynamics that such a choice is always possible.

eliminated by taking the origin as the position of zero potential energy. Thus we have

$$U = \beta_2 q^2 + \beta_3 q^3 + \cdots \qquad \text{(XXVII}')$$

But since the vibrations are supposed to be small, q remains a small quantity during the motion. Therefore the higher powers of q are negligible compared with q^2. Thus neglecting the higher terms we obtain the following expression for the potential energy of the system.

$$U = \tfrac{1}{2}\,\beta q^2, \qquad \text{(XXVIII)}$$

where $\frac{1}{2}\,\beta = \beta_2$.

The kinetic energy, on the other hand, takes the form

$$T = \tfrac{1}{2}\,\alpha \dot{q}^2, \qquad \text{(XXIX)}$$

where α is a constant and $\dot{q} = \dfrac{dq}{dt}$. But since the system is conservative the sum of its dynamical energy remains constant. Therefore

$$\begin{aligned} E &= T + U \\ &= \tfrac{1}{2}\,\alpha\dot{q}^2 + \tfrac{1}{2}\,\beta q^2. \end{aligned} \qquad \text{(XXX)}$$

Differentiating both sides of the last equation with respect to the time,

$$\alpha\ddot{q} + \beta q = 0, \qquad \text{(XXXI)}$$

which is the differential equation of simple harmonic motion. Therefore we have

$$q = a \sin \sqrt{\frac{\beta}{\alpha}}\,(t + t_0) \qquad \text{(XXXII)}$$

and

$$P = 2\pi \sqrt{\frac{\alpha}{\beta}}. \qquad \text{(XXXIII)}$$

Hence the main part of Lagrange's method consists of selecting the coördinate which defines the position of the system in such a way as to make the expressions for the kinetic and potential energies of the forms

$$\begin{aligned} T &= \tfrac{1}{2}\,\alpha\dot{q}^2, \\ U &= \tfrac{1}{2}\,\beta q^2. \end{aligned}$$

ILLUSTRATIVE EXAMPLES.

1. A weight which is suspended by means of a helical spring vibrates in the gravitational field of the earth. Find the expression for the period, taking the mass of the spring into account.

Let m = mass of the suspended body.
m' = mass of the spring.
ρ = mass per unit length of the spring.
L = length of the spring before the body is suspended.
D = increase in the length of the spring due to the weight of the suspended body.
a = the distance through which the body is pulled down in order to start the vibration.

In Fig. 147 let O denote the position of equilibrium, A the lowest position, and B any position of the body. The coördinate in terms of which we want to express the energy of the system must vanish at the position of equilibrium. Therefore we will define the position of the suspended body in terms of its distance from the position of equilibrium. The distance will be considered as positive when measured downwards. Let q denote this distance then the kinetic energy of the suspended body equals $\frac{1}{2} m\dot{q}^2$. In order to express the kinetic energy of the spring in terms of this coördinate let x denote the distance of an element of the spring from the point of suspension. Then the kinetic energy of the entire spring is

$$
\begin{aligned}
T_s &= \frac{1}{2}\int_0^{m'} \dot{x}^2\, dm \\
&= \frac{1}{2}\int_0^{L} \frac{x^2}{L^2}\dot{q}^2 \cdot \rho\, dx \qquad \left(\frac{\dot{x}}{\dot{q}} = \frac{x}{L}\right) \\
&= \frac{1}{2}\,\frac{\rho\dot{q}^2}{L^2}\int_0^{L} x^2\, dx \\
&= \frac{1}{2}\,\frac{\rho L}{3}\,\dot{q}^2 \\
&= \frac{1}{2}\,\frac{m'}{3}\,\dot{q}^2.
\end{aligned}
$$

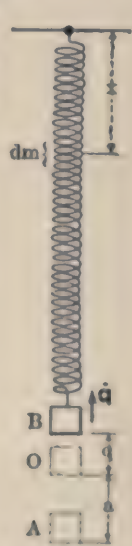


FIG. 147.

Hence the kinetic energy of the entire system is

$$T = \frac{1}{2}\left(m + \frac{m'}{3}\right)\dot{q}^2.$$

By Hooke's law the force which produces the extension of the string is a harmonic force, that is, if Q denotes the force then $Q = -kq$, where k is a constant. Therefore the potential energy of the system is

$$\begin{aligned} U &= -\int_0^q Q\,dq \\ &= k\int_0^q q\,dq \\ &= \tfrac{1}{2}kq^2. \end{aligned}$$

But $Q = mg$ when $q = -D$. Therefore $mg = kD$, or $k = \frac{mg}{D}$. Making this substitution in the expression for the potential energy we obtain

$$U = \frac{1}{2}\frac{mg}{D}q^2.$$

Therefore the total energy of the system is

$$\begin{aligned} E &= T + U \\ &= \frac{1}{2}\left(m + \frac{m'}{3}\right)\dot{q}^2 + \frac{1}{2}\frac{mg}{D}q^2. \end{aligned}$$

Differentiating the last equation with respect to t we obtain

$$\left(m + \frac{m'}{3}\right)\ddot{q} + \frac{mg}{D}q = 0,$$

which is the equation of simple harmonic motion. Therefore

$$q = a\sin\sqrt{\frac{mg}{\left(m + \frac{m'}{3}\right)D}}\,(t + t_0)$$

and

$$P = 2\pi\sqrt{\left(1 + \frac{m'}{3\,m}\right)\frac{D}{g}}.$$

It will be observed that, as in the case of every true harmonic motion, the period is not affected by the amplitude.

When the mass of the spring is negligible compared with that of the suspended weight the last two equations become

$$q = a\sin\sqrt{\frac{g}{D}}\,(t + t_0),$$

$$P = 2\pi\sqrt{\frac{D}{g}}.$$

Therefore in this case the length of the equivalent simple pendulum equals the stretch in the length of the spring produced by suspending the weight.

2. A particle of mass m is attached to the middle point of a stretched elastic string of natural length L, modulus of elasticity λ, and of negligible

mass. Find the period with which the particle will vibrate when displaced along the string.

Let L' be the stretched length of the string, A the area of its cross-section, q the distance of the particle from its position of equilibrium, and T_1 and T_2 the tensile forces of the two parts of the string. Then by Hooke's law we have

$$\frac{T_1}{A} = \lambda \frac{\left(\frac{L'}{2} + q\right) - \frac{L}{2}}{\frac{L}{2}} = \lambda \frac{L' - L + 2q}{L},$$

$$\frac{T_2}{A} = \lambda \frac{\left(\frac{L'}{2} - q\right) - \frac{L}{2}}{\frac{L}{2}} = \lambda \frac{L' - L - 2q}{L}.$$

Therefore the resultant force on the particle is

$$Q = T_2 - T_1 = -\frac{4A\lambda}{L} q = -\frac{4\lambda'}{L} q$$

where $\lambda' = A\lambda$. Hence the potential energy equals

$$U = -\int_0^q Q\,dq = \frac{2\lambda'}{L} q^2.$$

But since the kinetic energy is given by

$$T = \frac{1}{2} m\dot{q}^2$$

we obtain

$$E = \frac{1}{2} m\dot{q}^2 + \frac{2\lambda'}{L} q^2$$

for the total energy of the system. Differentiating the last equation we get

$$m\ddot{q} + \frac{4\lambda'}{L} q = 0,$$

which gives

$$q = a \sin \sqrt{\frac{4\lambda'}{mL}}\,(t + t_0)$$

and

$$P = \pi \sqrt{\frac{mL}{\lambda'}}.$$

3. A cylinder performs small oscillations inside of a fixed cylinder. Find the period of the motion, supposing the contact between the cylinders to be rough enough to prevent sliding.

Let m be the mass of the vibrating cylinder and a and b the radii of the vibrating and the fixed cylinders, respectively. Then at any instant

$$T = \tfrac{1}{2} I\omega^2,$$

where T denotes the kinetic energy of the vibrating cylinder, I its moment of inertia about the element of contact and ω its angular velocity. But

$$I = \tfrac{3}{2} ma^2,$$

and
$$\omega = \frac{v}{a} = (b - a)\dot{\theta},$$

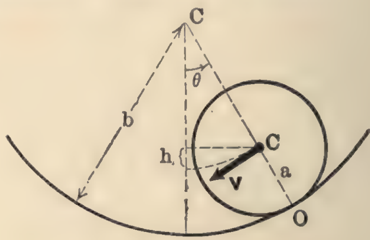


FIG. 148.

where v is the linear velocity of the axis of the moving cylinder and $\dot{\theta}$ its angular velocity. Therefore

$$T = \tfrac{3}{4} m (b - a)^2 \dot{\theta}^2.$$

On the other hand we have the following expressions for the potential energy:

$$\begin{aligned} U &= mgh \\ &= mg(b - a)(1 - \cos\theta) \\ &= mg(b - a)\left[1 - \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right)^*\right]. \end{aligned}$$

Since θ is supposed to remain small all the time, it is permissible to neglect the higher terms of θ in the last expression for U. Therefore we have

$$U = \tfrac{1}{2} mg(b - a)\theta^2.$$

Thus both T and U are expressed in forms which are adapted to the application of Lagrange's method.

The total energy of the system is

$$E = \tfrac{3}{4} m (b - a)^2 \dot{\theta}^2 + \tfrac{1}{2} mg(b - a)\theta^2.$$

Differentiating the last equation with respect to the time we obtain

$$3(b - a)\ddot{\theta} + 2g\theta = 0.$$

Therefore

$$\theta = \alpha \sin \sqrt{\frac{2g}{3(b - a)}}\,(t + t_0),$$

and
$$P = \pi \sqrt{\frac{6(b - a)}{g}}.$$

* The expansion is carried out by Maclaurin's Theorem. See Appendix AI.

When the contact is smooth we have

$$T = \tfrac{1}{2}\, m\, (b - a)^2 \dot{\theta}^2,$$

and

$$U = \tfrac{1}{2}\, mg\, (b - a)\, \theta^2.$$

Therefore

$$\theta = \alpha \sin \sqrt{\frac{g}{b - a}}\, (t + t_0),$$

and

$$P = 2\pi \sqrt{\frac{b - a}{g}}.$$

Thus the length of the equivalent simple pendulum is $(b - a)$ when the contact is smooth and $\frac{3\,(b - a)}{2}$ when it is rough.

PROBLEMS.

1. A butcher's balance is elongated 1 inch when a weight of 4 pounds is placed in the pan. If the spring of the balance weighs 5 ounces, find the error introduced by neglecting the mass of the spring in calculating the period of oscillation.

2. Find the expression for the period of vibration of mercury in a U-tube.

3. If in the illustrative problem on p. 329 the particle divides the string in the ratio of 1 to n, show that the period is $P = 2\pi \sqrt{\frac{n-1}{n^2} \cdot \frac{mL}{\lambda'}}$.

4. Find the period of vibration of a homogeneous hemisphere which performs small oscillations upon a horizontal plane which is rough enough to prevent sliding.

5. Find the period of vibration of a homogeneous sphere which makes small oscillations in a fixed rough sphere.

6. A particle of mass m is attached to a point on a smooth horizontal table by means of a spring of natural length L. If the particle is pulled so that the spring is stretched to twice its natural length and then let go, show that it will vibrate with a period $P = 2\,(\pi + 2) \sqrt{\frac{mL}{T}}$, where T is the force necessary to stretch the spring to twice its natural length. The mass of the spring is negligible.

7. Two masses m_1 and m_2 are connected by a spring of negligible mass. The modulus of elasticity of the spring is such that when m_1 is fixed m_2 makes n vibrations per second. Show that when m_2 is fixed m_1 makes $n \sqrt{\frac{m_2}{m_1}}$ vibrations per second.

8. In the preceding problem suppose both of the particles to be free and show that they make $n\sqrt{\frac{m_1 + m_2}{m_1}}$ vibrations per second.

9. A string which connects two particles of equal mass passes through a small hole in a smooth horizontal table. One of the particles hangs vertically while the other, which is on the table at a distance D from the hole, is given a velocity $\sqrt{g\,D}$ in a direction perpendicular to the string. Show that the suspended particle will be in equilibrium and that if it is slightly disturbed it will vibrate with a period of $2\,\pi\sqrt{\frac{2\,D}{3\,g}}$.

10. The piston of a cylinder, which is in a vertical position, is in equilibrium under the action of its weight and the upward pressure of the gas in the cylinder. Show that when the cylinder is given a small displacement it will vibrate with a period equal to $2\,\pi\sqrt{\frac{h}{g}}$, where h is the height of the piston above the base of the cylinder when the former is at its equilibrium position. Assume Boyle's law to hold.

11. In illustrative problem 2 (p. 328) take the mass of the string into account and obtain the expression for the period of vibration.

12. In problem 6 take the mass of the spring into account and obtain an expression for the period.

13. In problem 7 take the mass of the spring into account and find the expression for the period of vibrations.

14. In problem 8 take the mass of the spring into account and find the expression for the period.

15. A particle is placed at the center of a smooth horizontal table; two particles of the same mass as the first one are suspended by means of strings of negligible mass, each of which passes over a smooth pulley at the middle point of one of the edges of the table and is attached to the first particle. The particle at the center is given a small displacement at right angles to the strings. Show that it performs small oscillations with a period of $2\,\pi\sqrt{\frac{a}{g}}$, where a is the distance between the two pulleys.

16. A particle rests at the center of a square table which is smooth and horizontal. Four particles are suspended by means of strings each of which passes over an edge of the table and is connected to the particle on the table. Find the period with which the system will vibrate when the particle which is on the table is displaced along one of the strings. The particles have equal mass. Neglect the mass of the strings.

17. A particle is in equilibrium at a point midway between two centers of attraction, which attract the particle with forces proportional to the distance. Show that if the particle is displaced toward one of the centers it will vibrate with a period of $\frac{2\pi}{\sqrt{K+K'}}$, where K and K' are the forces which a unit mass would experience when placed at a unit distance from each center of force.

APPENDIX A.

MATHEMATICAL FORMULÆ.

I. BINOMIAL THEOREM.

$$(a + x)^n = a^n + \frac{n}{1!} a^{n-1} x + \frac{n(n-1)}{2!} a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}$$
$$a^{n-3}x^3 + \cdots$$
$$\doteq a^n\left(1 + \frac{x}{a}\right). \quad \left[\begin{array}{l}\text{When } x \ll a, \text{ and consequently} \\ \frac{x^2}{a^2}, \frac{x^3}{a^3}, \text{ etc.,} \ll x.\end{array}\right]$$

Applying this theorem to $(1 \pm x)^{-1}$ we obtain

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$
$$\doteq 1 - x. \quad \left[\begin{array}{l}\text{When } x \ll 1, \text{ and consequently} \\ x^2, x^3, \text{ etc.,} \ll x.\end{array}\right]$$
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$
$$\doteq 1 + x. \quad \left[\begin{array}{l}\text{When } x \ll 1, \text{ and consequently} \\ x^2, x^3, \text{ etc.,} \ll x.\end{array}\right]$$

II. QUADRATIC FORMULA.

If x satisfies the quadratic equation $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

III. LOGARITHMIC RELATIONS.

(a) $\log ab = \log a + \log b$.

(b) $\log a^n = n \log a$. [This formulæ may be obtained from (a) by letting $b = a$, a^2, etc., until $ab = a^n$.]

(c) $\log \frac{a}{b} = \log a - \log b$. [This follows immediately from (a) and (b).]

(d) $\log 1 = 0$. [This is obtained by letting $b = a$ in (c).]

IV. TRIGONOMETRIC RELATIONS.

(a) $\sin^2 x + \cos^2 x = 1$.

(b) $1 + \tan^2 x = \sec^2 x$.

(c) $\sin(x \pm y) = \sin x \cos y + \cos x \sin(\pm y)$.

(d) $\cos(x \pm y) = \cos x \cos y - \sin x \sin(\pm y)$.

(e) $\tan(x \pm y) = \dfrac{\tan x + \tan(\pm y)}{1 - \tan x \tan(\pm y)}$.

(f) $\sin 2x = 2 \sin x \cos x$.

(g) $\cos 2x = \cos^2 x - \sin^2 x$.

(h) $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$.

[(f), (g), and (h) are obtained by letting $y = x$ in (c), (d), and (e).]

(i) $\sin x = 2 \sin \dfrac{x}{2} \cos \dfrac{x}{2}$.

(j) $\cos x = \cos^2 \dfrac{x}{2} - \sin^2 \dfrac{x}{2}$.

(k) $\tan x = \dfrac{2 \tan \dfrac{x}{2}}{1 - \tan^2 \dfrac{x}{2}}$.

[These may be obtained by replacing x by $\dfrac{x}{2}$ in (f), (g), and (h).]

(l) $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.

(m) $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.

[These may be obtained easily from (g).]

(n) $\sin^2 \dfrac{x}{2} = \dfrac{1}{2}(1 - \cos x)$.

(o) $\cos^2 \dfrac{x}{2} = \dfrac{1}{2}(1 + \cos x)$.

[These are obtained by replacing x by $\dfrac{x}{2}$ in (l) and (m).]

Angle between two lines.

(p) $\cos\theta = \cos\alpha \cos\alpha' + \cos\beta \cos\beta' + \cos\gamma \cos\gamma'$.

V. MACLAURIN'S THEOREM.

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

VI. IMPORTANT FUNCTIONS EXPRESSED AS POWER SERIES.

The following expansions are carried out by Maclaurin's theorem.

(a) $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$

$\doteq 1 + x$. [When $x \ll 1$, and consequently x^2, x^3, etc., $\ll x$.]

(b) $e^{ix} = 1 + \dfrac{ix}{1!} - \dfrac{x^2}{2!} - \dfrac{ix^3}{3!} + \dfrac{x^4}{4!} + \cdots$

(c) $e^{-ix} = 1 - \dfrac{ix}{1!} - \dfrac{x^2}{2!} + \dfrac{ix^3}{3!} + \dfrac{x^4}{4!} - \cdots$

(d) $\sin x = \dfrac{x}{1!} - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$

$\doteq x$. [When $x \ll 1$, and consequently x^2, x^3, etc., $\ll x$.]

(e) $\cos x = \frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, [0! = 1]$

$\doteq 1.$ [When $x \ll 1$, and consequently x^2, x^3, etc., $\ll x$.]

(f) $\log(1+x) = \frac{x}{1} - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots$, for $-1 < x < 1$

$= x.$ [When $x \ll 1$, and consequently x^2, x^3, etc., $\ll x$.]

VII. RELATIONS WHICH CONNECT EXPONENTIAL FUNCTIONS WITH CIRCULAR FUNCTIONS.

(a) $e^{ix} = \cos x + i \sin x.$

(b) $e^{-ix} = \cos x - i \sin x.$

[These are called De Moivre's Theorems and are obtained by comparing series (b) and (c) of VI with series (d) and (e) of the same group.]

(c) $\sin x = \frac{e^{ix} - e^{-ix}}{2i}.$ [This relation is obtained by subtracting (b) from (a).]

(d) $\cos x = \frac{e^{ix} + e^{-ix}}{2}.$ [This relation is obtained by adding (b) to (a).]

VIII. HYPERBOLIC FUNCTIONS.

(a) $\sinh x = i \sin(ix).$

(b) $\cosh x = \cos(ix).$

These are the definitions of the hyperbolic sine and the hyperbolic cosine. Replacing x by ix in equations (c) and (d) of group VII we obtain the following relations between hyperbolic and exponential functions:

(c) $\sinh x = \frac{e^x - e^{-x}}{2},$

(d) $\cosh x = \frac{e^x + e^{-x}}{2}.$

Squaring equation (c) and subtracting it from the square of equation (d) we obtain

(e) $\cosh^2 x - \sinh^2 x = 1.$

IX. AVERAGE VALUE.

The average value of $y = f(x)$ in the interval between $x = x_1$ and $x = x_2$ is given by

$$\bar{y} = \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} y\, dx.$$

APPENDIX B.

MATHEMATICAL TABLES.

Logarithms of Numbers.

N.	0	1	2	3	4	5	6	7	8	9
0	....	0000	3010	4771	6021	6990	7782	8451	9031	9542
1	0000	0414	0792	1139	1461	1761	2041	2304	2553	2788
2	3010	3222	3424	3617	3802	3979	4150	4314	4472	4624
3	4771	4914	5051	5185	5315	5441	5563	5682	5798	5911
4	6021	6128	6232	6335	6435	6532	6628	6721	6812	6902
5	6990	7076	7160	7243	7324	7404	7482	7559	7634	7709
6	7782	7853	7924	7993	8062	8129	8195	8261	8325	8388
7	8451	8513	8573	8633	8692	8751	8808	8865	8921	8976
8	9031	9085	9138	9191	9243	9294	9345	9395	9445	9494
9	9542	9590	9638	9685	9731	9777	9823	9868	9912	9956
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5900
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
N.	0	1	2	3	4	5	6	7	8	9

P.P.

	22	21	20	19	18	17	16	15	14	13	12	11	9	8
1	2.2	2.1	2.0	1.9	1.8	1.7	1.6	1.5	1.4	1.3	1.2	1.1	0.9	0.8
2	4.4	4.2	4.0	3.8	3.6	3.4	3.2	3.0	2.8	2.6	2.4	2.2	1.8	1.6
3	6.6	6.3	6.0	5.7	5.4	5.1	4.8	4.5	4.2	3.9	3.6	3.3	2.7	2.4
4	8.8	8.4	8.0	7.6	7.2	6.8	6.4	6.0	5.6	5.2	4.8	4.4	3.6	3.2
5	11.0	10.5	10.0	9.5	9.0	8.5	8.0	7.5	7.0	6.5	6.0	5.5	4.5	4.0
6	13.2	12.6	12.0	11.4	10.8	10.2	9.6	9.0	8.4	7.8	7.2	6.6	5.4	4.8
7	15.4	14.7	14.0	13.3	12.6	11.9	11.2	10.5	9.8	9.1	8.4	7.7	6.3	5.6
8	17.6	16.8	16.0	15.2	14.4	13.6	12.8	12.0	11.2	10.4	9.6	8.8	7.2	6.4
9	19.8	18.9	18.0	17.1	16.2	15.3	14.4	13.5	12.6	11.7	10.8	9.9	8.1	7.2

Logarithms of Numbers.

N.	0	1	2	3	4	5	6	7	8	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
100	0000	0004	0009	0013	0017	0022	0026	0030	0035	0039
N.	0	1	2	3	4	5	6	7	8	9

P. P.

	9	8	7	6	5	4
1	0.9	0.8	0.7	0.6	0.5	0.4
2	1.8	1.6	1.4	1.2	1.0	0.8
3	2.7	2.4	2.1	1.8	1.5	1.2
4	3.6	3.2	2.8	2.4	2.0	1.6
5	4.5	4.0	3.5	3.0	2.5	2.0
6	5.4	4.8	4.2	3.6	3.0	2.4
7	6.3	5.6	4.9	4.2	3.5	2.8
8	7.2	6.4	5.6	4.8	4.0	3.2
9	8.1	7.2	6.3	5.4	4.5	3.6

$\pi = 3.1416$
$\log \pi = 0.4971$
$e = 2.7183$
$\log e = 0.4343$
$y = \log_e x$
$= 2.292 \log_{10} x$

Trigonometric Functions.

0° – 22°.5.

Degree.	Sine.	Tangent.	Cotangent.	Cosine.	
0.0	0.0000	0.0000	∞	1.0000	90.0
0.5	0.0087	0.0087	114.5887	1.0000	89.5
1.0	0.0175	0.0175	57.2900	0.9998	89.0
1.5	0.0262	0.0262	38.1885	0.9997	88.5
2.0	0.0349	0.0349	28.6363	0.9994	88.0
2.5	0.0436	0.0437	22.9038	0.9990	87.5
3.0	0.0523	0.0524	19.0811	0.9986	87.0
3.5	0.0610	0.0612	16.3499	0.9981	86.5
4.0	0.0698	0.0699	14.3007	0.9976	86.0
4.5	0.0785	0.0787	12.7062	0.9969	85.5
5.0	0.0872	0.0875	11.4301	0.9962	85.0
5.5	0.0958	0.0963	10.3854	0.9954	84.5
6.0	0.1045	0.1051	9.5144	0.9945	84.0
6.5	0.1132	0.1139	8.7769	0.9936	83.5
7.0	0.1219	0.1228	8.1443	0.9925	83.0
7.5	0.1305	0.1317	7.5958	0.9914	82.5
8.0	0.1392	0.1405	7.1154	0.9903	82.0
8.5	0.1478	0.1495	6.6912	0.9890	81.5
9.0	0.1564	0.1584	6.3138	0.9877	81.0
9.5	0.1650	0.1673	5.9758	0.9863	80.5
10.0	0.1736	0.1763	5.6713	0.9848	80.0
10.5	0.1822	0.1853	5.3955	0.9833	79.5
11.0	0.1908	0.1944	5.1446	0.9816	79.0
11.5	0.1994	0.2035	4.9152	0.9799	78.5
12.0	0.2079	0.2126	4.7046	0.9781	78.0
12.5	0.2164	0.2217	4.5107	0.9763	77.5
13.0	0.2250	0.2309	4.3315	0.9744	77.0
13.5	0.2334	0.2401	4.1653	0.9724	76.5
14.0	0.2419	0.2493	4.0108	0.9703	76.0
14.5	0.2504	0.2586	3.8667	0.9681	75.5
15.0	0.2588	0.2679	3.7321	0.9659	75.0
15.5	0.2672	0.2773	3.6059	0.9636	74.5
16.0	0.2756	0.2867	3.4874	0.9613	74.0
16.5	0.2840	0.2962	3.3759	0.9588	73.5
17.0	0.2924	0.3057	3.2709	0.9563	73.0
17.5	0.3007	0.3153	3.1716	0.9537	72.5
18.0	0.3090	0.3249	3.0777	0.9511	72.0
18.5	0.3173	0.3346	2.9887	0.9483	71.5
19.0	0.3256	0.3443	2.9042	0.9455	71.0
19.5	0.3338	0.3541	2.8239	0.9426	70.5
20.0	0.3420	0.3640	2.7475	0.9397	70.0
20.5	0.3502	0.3739	2.6746	0.9367	69.5
21.0	0.3584	0.3839	2.6051	0.9336	69.0
21.5	0.3665	0.3939	2.5386	0.9304	68.5
22.0	0.3746	0.4040	2.4751	0.9272	68.0
22.5	0.3827	0.4142	2.4142	0.9239	67.5
	Cosine.	Cotangent.	Tangent.	Sine.	Degree.

67°.5 – 90°.

Trigonometric Functions.
22°.5 – 45°.

Degree.	Sine.	Tangent.	Cotangent.	Cosine.	
22.5	0.3827	0.4142	2.4142	0.9239	67.5
23.0	0.3907	0.4245	2.3559	0.9205	67.0
23.5	0.3987	0.4348	2.2998	0.9171	66.5
24.0	0.4067	0.4452	2.2460	0.9135	66.0
24.5	0.4147	0.4557	2.1943	0.9100	65.5
25.0	0.4226	0.4663	2.1445	0.9063	65.0
25.5	0.4305	0.4770	2.0965	0.9026	64.5
26.0	0.4384	0.4877	2.0503	0.8988	64.0
26.5	0.4462	0.4986	2.0057	0.8949	63.5
27.0	0.4540	0.5095	1.9626	0.8910	63.0
27.5	0.4617	0.5206	1.9210	0.8870	62.5
28.0	0.4695	0.5317	1.8807	0.8829	62.0
28.5	0.4772	0.5430	1.8418	0.8788	61.5
29.0	0.4848	0.5543	1.8040	0.8746	61.0
29.5	0.4924	0.5658	1.7675	0.8704	60.5
30.0	0.5000	0.5774	1.7321	0.8660	60.0
30.5	0.5075	0.5890	1.6977	0.8616	59.5
31.0	0.5150	0.6009	1.6643	0.8572	59.0
31.5	0.5225	0.6128	1.6319	0.8526	58.5
32.0	0.5299	0.6249	1.6003	0.8480	58.0
32.5	0.5373	0.6371	1.5697	0.8434	57.5
33.0	0.5446	0.6494	1.5399	0.8377	57.0
33.5	0.5519	0.6619	1.5108	0.8339	56.5
34.0	0.5592	0.6745	1.4826	0.8290	56.0
34.5	0.5664	0.6873	1.4550	0.8241	55.5
35.0	0.5739	0.7002	1.4281	0.8192	55.0
35.5	0.5807	0.7133	1.4019	0.8141	54.5
36.0	0.5878	0.7265	1.3764	0.8090	54.0
36.5	0.5948	0.7400	1.3514	0.8039	53.5
37.0	0.6018	0.7536	1.3270	0.7986	53.0
37.5	0.6088	0.7673	1.3032	0.7934	52.5
38.0	0.6157	0.7813	1.2799	0.7880	52.0
38.5	0.6225	0.7954	1.2572	0.7826	51.5
39.0	0.6293	0.8098	1.2349	0.7771	51.0
39.5	0.6361	0.8243	1.2131	0.7716	50.5
40.0	0.6428	0.8391	1.1918	0.7660	50.0
40.5	0.6494	0.8541	1.1708	0.7604	49.5
41.0	0.6561	0.8693	1.1504	0.7547	49.0
41.5	0.6626	0.8847	1.1303	0.7490	48.5
42.0	0.6691	0.9004	1.1106	0.7431	48.0
42.5	0.6756	0.9163	1.0913	0.7373	47.5
43.0	0.6820	0.9325	1.0724	0.7314	47.0
43.5	0.6884	0.9490	1.0538	0.7254	46.5
44.0	0.6947	0.9657	1.0355	0.7193	46.0
44.5	0.7009	0.9827	1.0176	0.7133	45.5
45.0	0.7071	1.0000	1.0000	0.7071	45.0
	Cosine.	Cotangent.	Tangent.	Sine.	Degree.

45° – 67°.5.

Exponential Functions.

x	e^x	e^{-x}	x	e^x	e^{-x}	x	e^x	e^{-x}
0.00	1.000	1.000	0.60	1.822	0.549	2.5	12.18	0.0821
.01	1.010	0.990	.62	1.859	0.538	.6	13.46	0.0743
.02	1.020	0.980	.64	1.896	0.527	.7	14.88	0.0672
.03	1.030	0.970	.66	1.935	0.517	.8	16.44	0.0608
.04	1.041	0.961	.68	1.974	0.507	.9	18.17	0.0550
.05	1.051	0.951	.70	2.014	0.497	3.0	20.09	0.0498
.06	1.062	0.942	.72	2.054	0.487	.1	22.20	0.0450
.07	1.073	0.932	.74	2.096	0.477	.2	24.53	0.0408
.08	1.083	0.923	.76	2.138	0.468	.3	27.11	0.0369
.09	1.094	0.914	.78	2.181	0.458	.4	29.96	0.0334
.10	1.105	0.905	.80	2.226	0.449	3.5	33.12	0.0302
.12	1.127	0.887	.82	2.270	0.440	.6	36.60	0.0273
.14	1.150	0.869	.84	2.316	0.432	.7	40.45	0.0247
.16	1.174	0.852	.86	2.336	0.423	.8	44.70	0.0224
.18	1.197	0.835	.88	2.411	0.415	.9	49.40	0.0202
.20	1.221	0.817	.90	2.460	0.407	4.0	54.60	0.0183
.22	1.246	0.803	.92	2.507	0.399	.1	60.34	0.0166
.24	1.271	0.787	.94	2.560	0.391	.2	66.69	0.0150
.26	1.297	0.771	.96	2.612	0.383	.3	73.70	0.0136
.28	1.323	0.756	.98	2.664	0.375	4.5	90.02	0.0111
.30	1.350	0.741	1.00	2.718	0.368	5.0	148.4	0.00674
.32	1.377	0.726	.1	3.004	0.333	6.0	403.4	0.00248
.34	1.405	0.712	.2	3.320	0.301	7.0	1097.	0.000912
.36	1.433	0.698	.3	3.669	0.273	8.0	2981.	0.000335
.38	1.462	0.684	.4	4.055	0.247	9.0	8103.	0.000123
.40	1.492	0.670	1.5	4.482	0.223	10	22026.	.0000454
.42	1.522	0.657	.6	4.953	0.202	$\pi/4$	2.193	0.4560
.44	1.553	0.644	.7	5.474	0.183	$\pi/3$	2.847	0.3513
.46	1.584	0.631	.8	6.050	0.165	$\pi/2$	4.810	0.2079
.48	1.616	0.619	.9	6.686	0.150	$3\pi/4$	10.55	0.0948
.50	1.649	0.607	2.0	7.389	0.135	π	23.14	0.0432
.52	1.682	0.595	.1	8.166	0.122	$3\pi/2$	111.3	0.00898
.54	1.716	0.583	.2	9.025	0.111	2π	535.5	0.00187
.56	1.751	0.571	.3	9.974	0.100	$5\pi/2$	2576.	0.000388
.58	1.786	0.560	.4	11.02	0.907	3π	12392	0.000081
.60	1.822	0.549	2.5	12.18	0.0821	4π	286751.	0.000003

$$y = \log_e x = \log_{10} x / \log_{10} e = 2.292 \log_{10} x.$$

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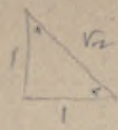
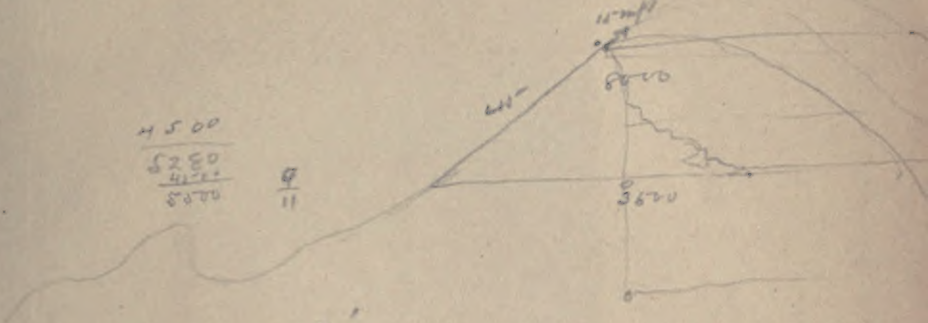


$\frac{1}{\sqrt{2}}$ $\frac{1}{2}$

4500
5280
4500
8000

$\frac{9}{11}$



R′

$$y = x \tan 45^\circ - \frac{g}{2V_0^2 \cos^2 45^\circ} x^2$$

$$-\frac{9}{11} = x - \frac{9}{4 \times 15^2} x^2$$

$$-\frac{4 \times 15 \times 15 \times 9}{11} = 4 \times 15 \times 15\, x - 9x^2$$

$$-\frac{4 \times 15 \times 15}{11} = 4 \times 5 \times 5\, x - x^2$$

$$x^2 - 4 \times 25\, x + \frac{4 \times 15 \times 15}{11} = 0$$

$$x = \frac{-4 \times 25 \pm \sqrt{16 \times 625 - \frac{16 \times 15^2}{121}}}{2}$$

$$\frac{-100 + \frac{1098}{11}}{2}$$

9.9.9

625
16
3750
625
10000
3600
1,20,64,0 | 1095

225
16
900
3600